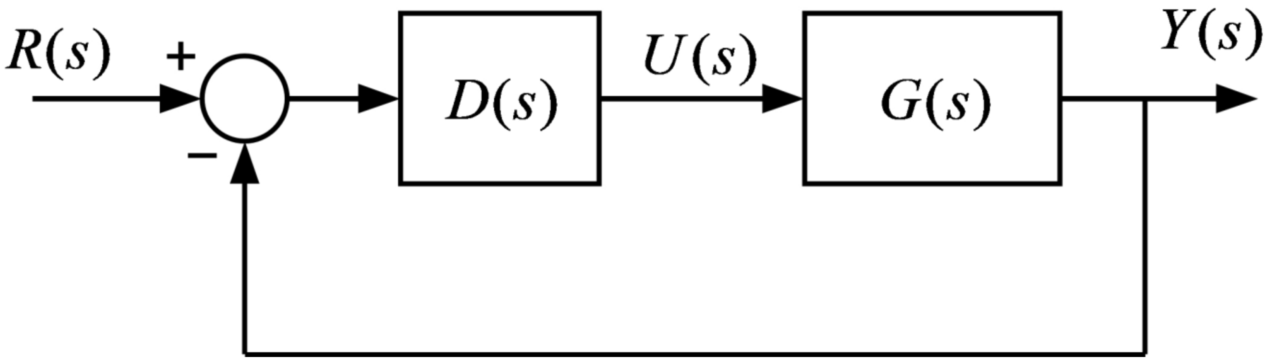
Review: Controller Design

in Frequency Domain

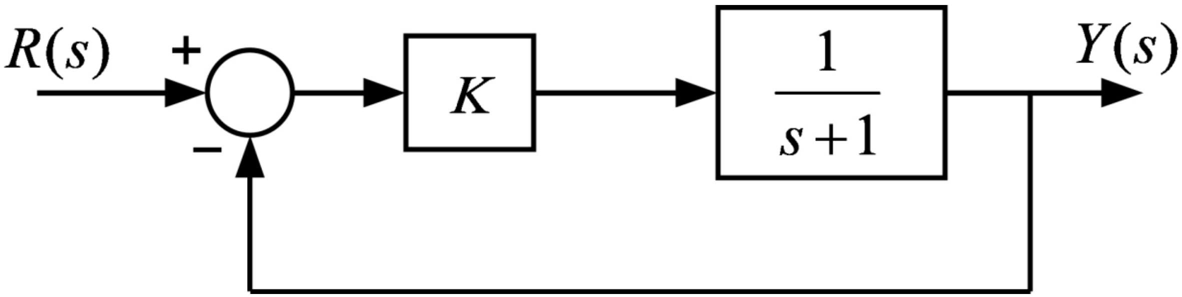
Feedback Control System



Closed-loop Transfer Function: $G_T(s) = \dfrac{D(s)G(s)}{1 + D(s)G(s)}$

Open-loop Transfer Function: $D(s)G(s)$

Root Locus

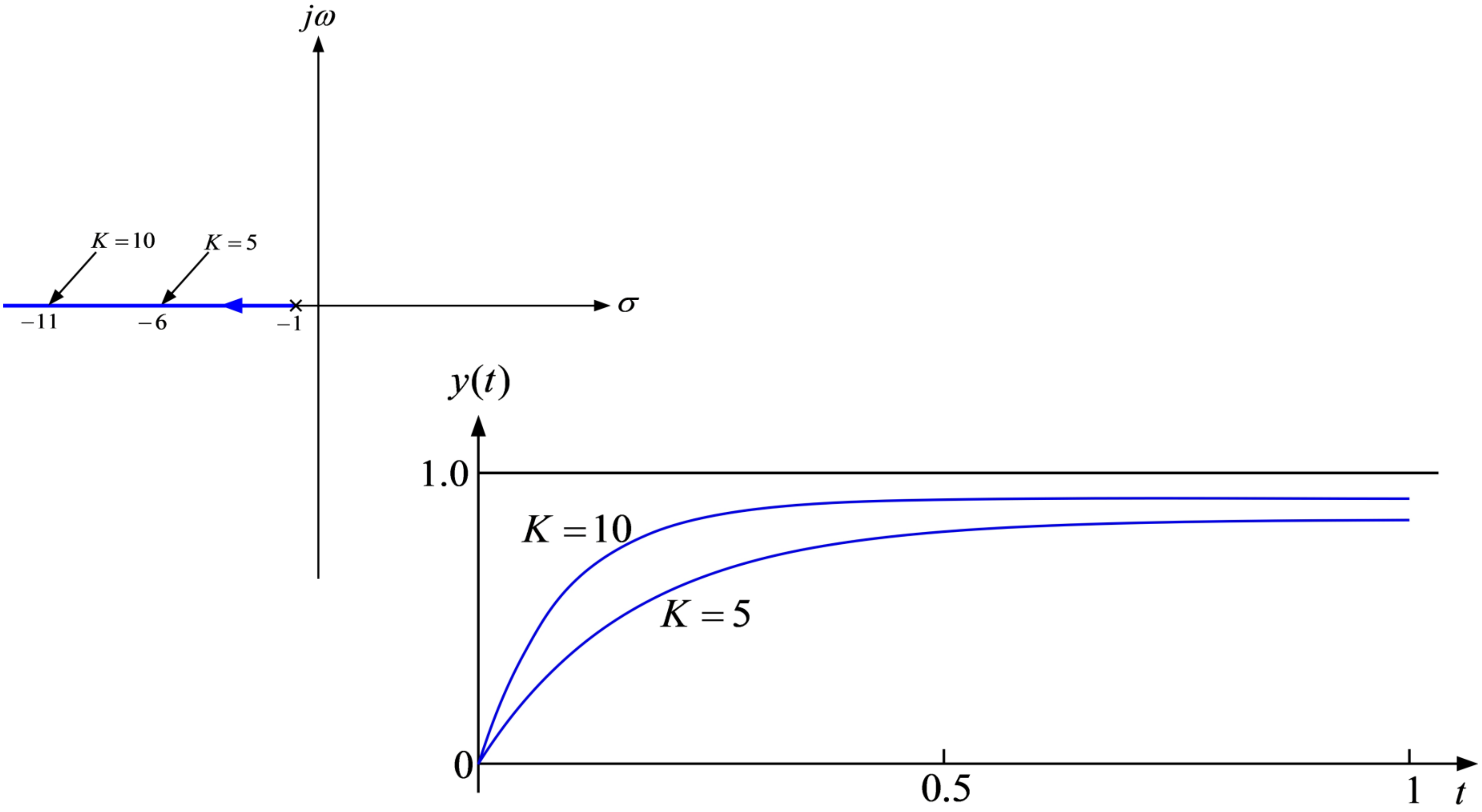


$$\frac{Y(s)}{R(s)} = \frac{\dfrac{K}{s+1}}{1+\dfrac{K}{s+1}} = \frac{K}{s+1+K}$$

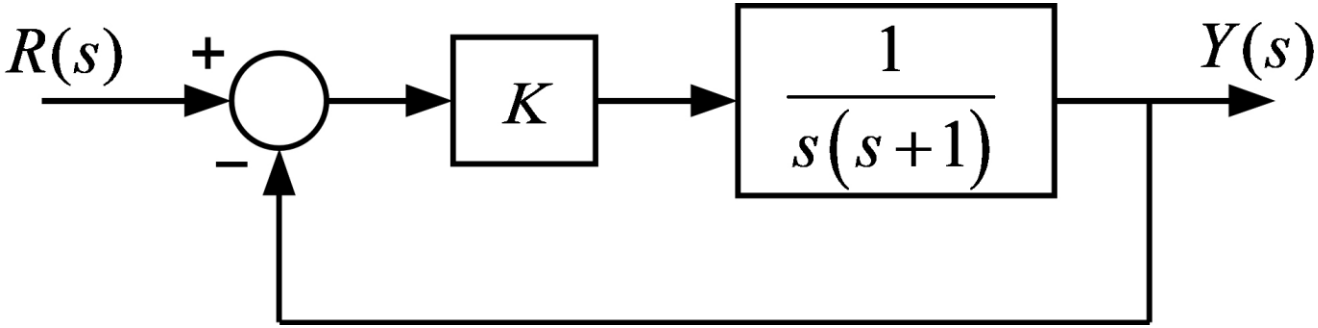
Characteristic Equation: $s+1+K=0$

Pole: $s=-1-K$

Root Locus



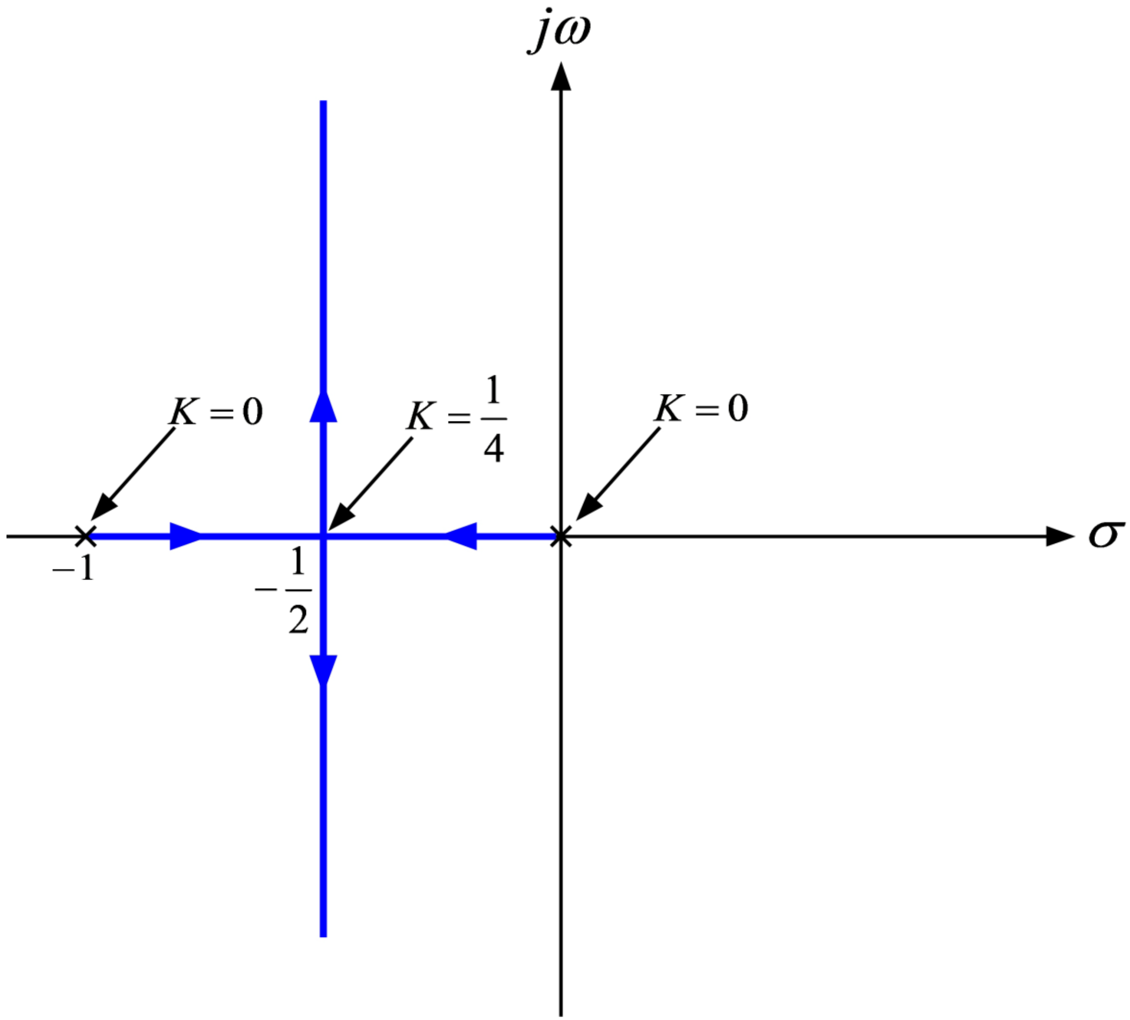
Root Locus



$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s(s+1)}}{1+\frac{K}{s(s+1)}} = \frac{K}{s^2+s+K}$$

$$s^2+s+K=0 \Rightarrow s = \frac{-1 \pm \sqrt{1-4K}}{2}$$

Root Locus



Frequency Response

$$\frac{Y(s)}{U(s)} = G(s) \qquad u(t) = A\sin\omega t$$

$$G(s) = \frac{b(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$Y(s) = G(s)U(s) = G(s)\frac{A\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \cdots + \frac{\alpha_n}{s-p_n} + \frac{\alpha_0}{s+j\omega} + \frac{\alpha_0^*}{s-j\omega}$$

$$y(t) = \alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \cdots + \alpha_n e^{p_n t} + \alpha_0 e^{-j\omega t} + \alpha_0^* e^{j\omega t}$$

Frequency Response

$$y(t) = \alpha_0 e^{-j\omega t} + \alpha_0^* e^{j\omega t}$$

$$\alpha_0 = G(s)\frac{A\omega}{s^2+\omega^2}\left(s+j\omega\right)\Big|_{s=-j\omega} = -\frac{A}{j2}G(-j\omega)$$

$$= -\frac{A}{j2}\left|G(j\omega)\right|e^{-j\angle G(j\omega)}$$

$$\alpha_0^* = G(s)\frac{A\omega}{s^2+\omega^2}\left(s-j\omega\right)\Big|_{s=j\omega} = \frac{A}{j2}G(j\omega)$$

$$= \frac{A}{j2}\left|G(j\omega)\right|e^{j\angle G(j\omega)}$$

Frequency Response

$$y(t) = -\frac{A}{j2}\left|G(j\omega)\right| e^{-j\angle G(j\omega)} e^{-j\omega t} + \frac{A}{j2}\left|G(j\omega)\right| e^{j\angle G(j\omega)} e^{j\omega t}$$

$$= A\left|G(j\omega)\right|\left(-\frac{e^{-j\left(\omega t+\angle G(j\omega)\right)}}{j2} + \frac{e^{j\left(\omega t+\angle G(j\omega)\right)}}{j2}\right)$$

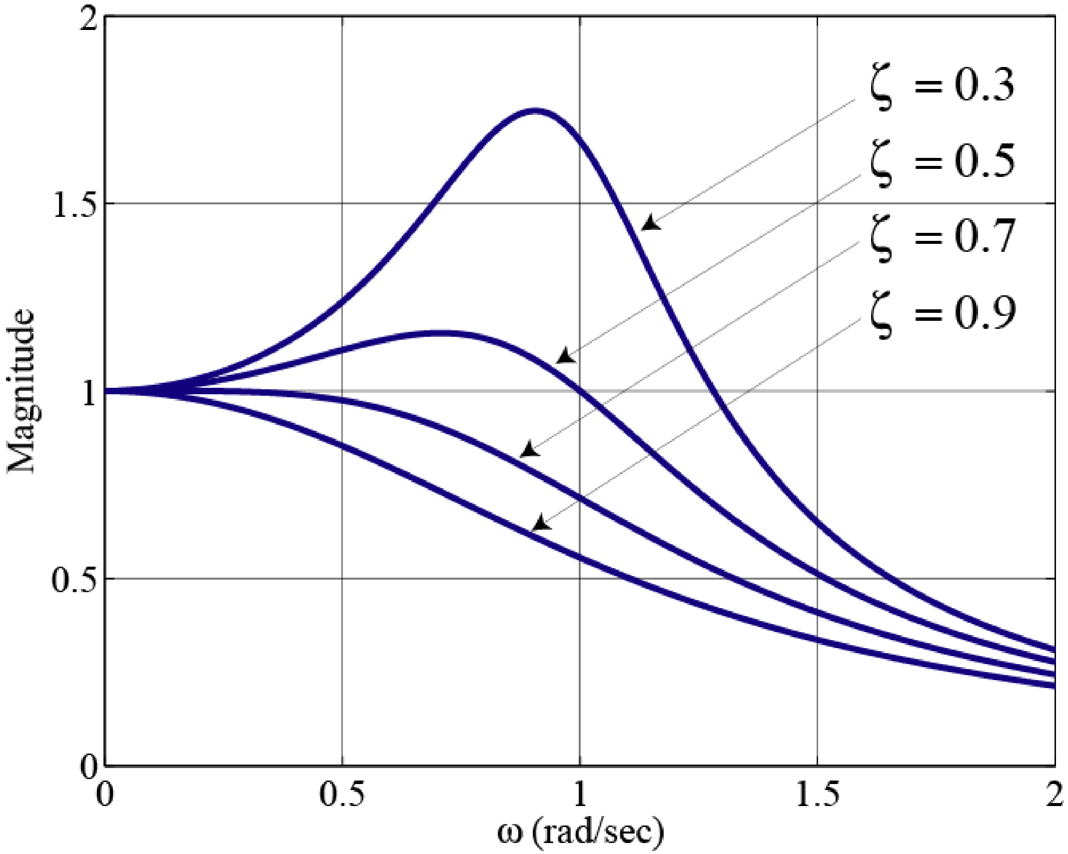
$$= A\left|G(j\omega)\right|\sin\left(\omega t + \angle G(j\omega)\right)$$

Frequency Response and Poles

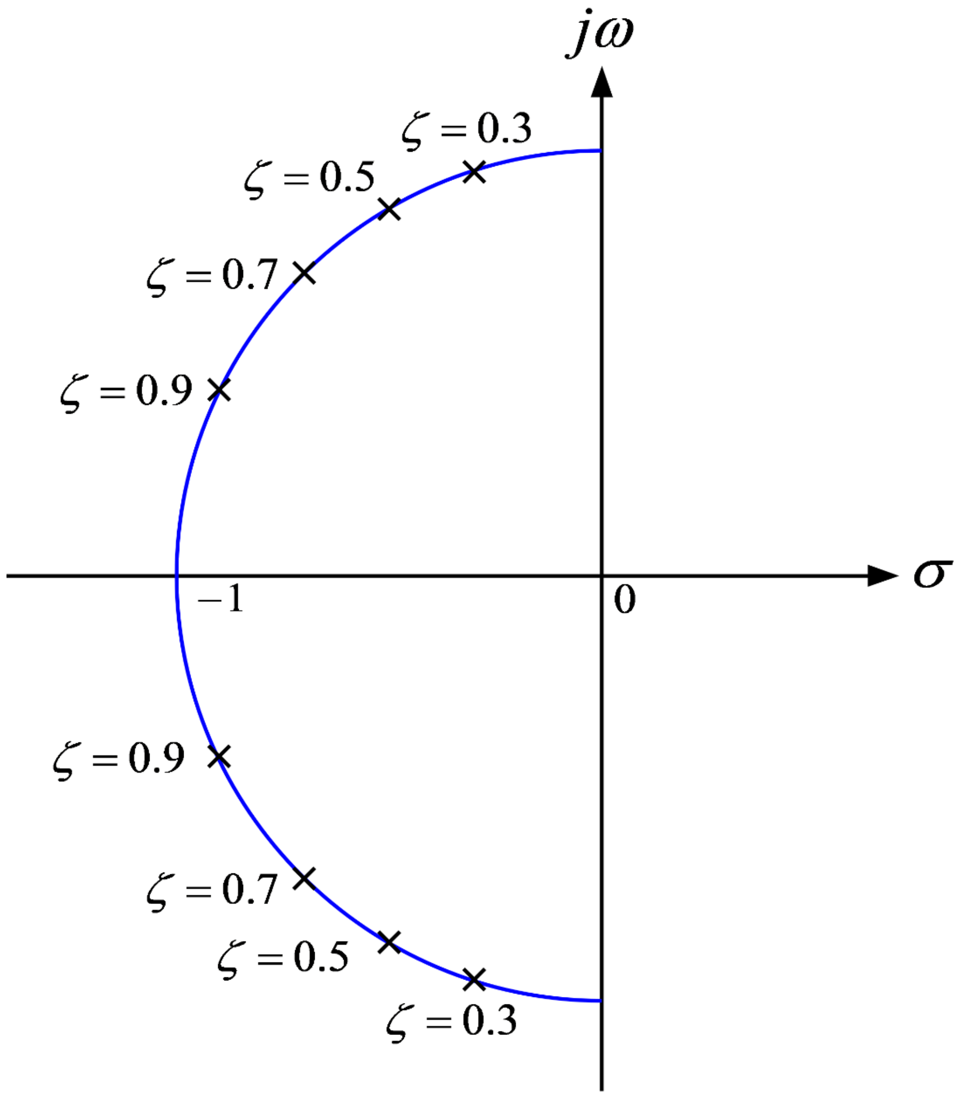
$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1}$$

$$\left|G(s)\right|$$

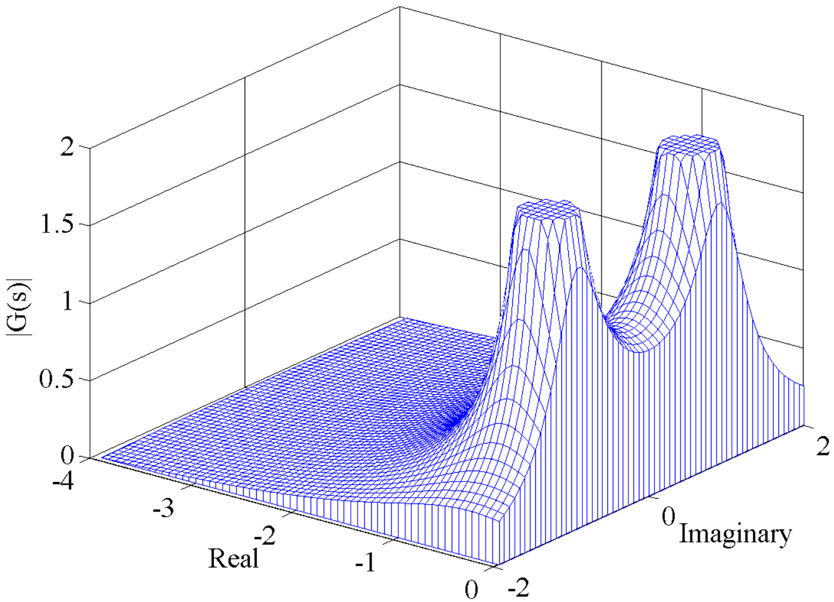
$$=\left|\frac{1}{\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1}\right|$$



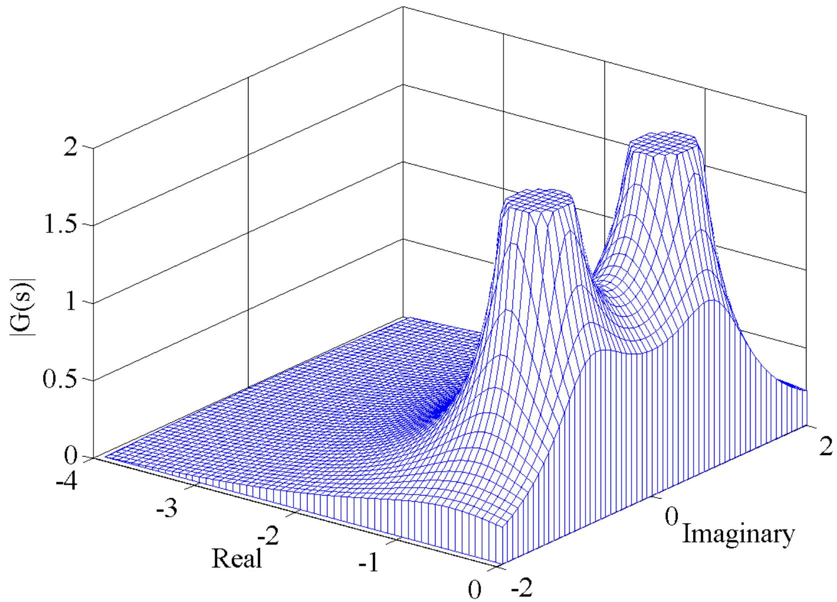
Pole Locations



$|G(s)|$

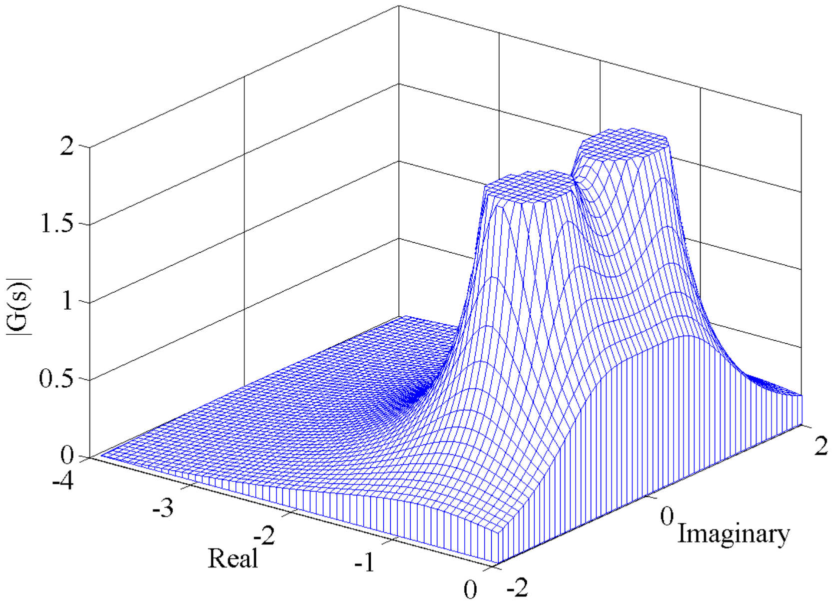


$\zeta = 0.3$

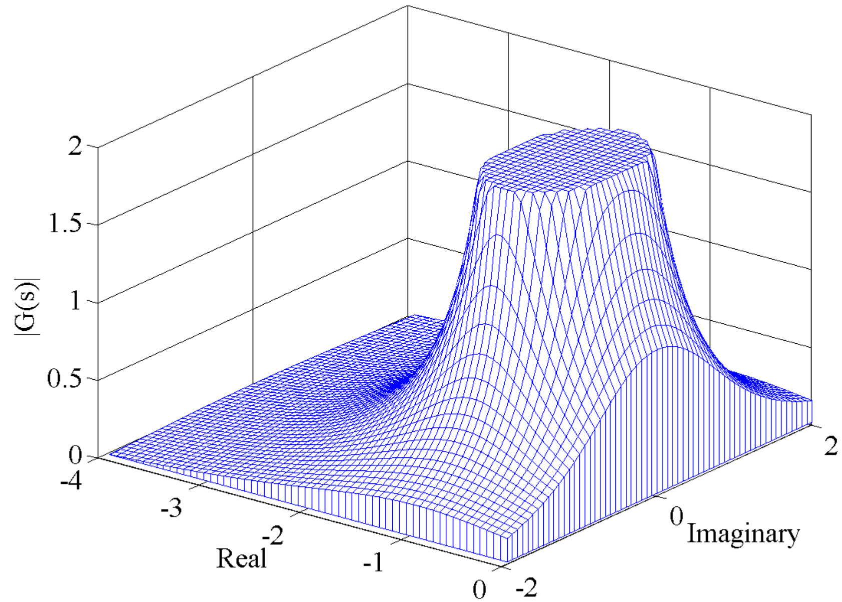


$\zeta = 0.5$

$|G(s)|$



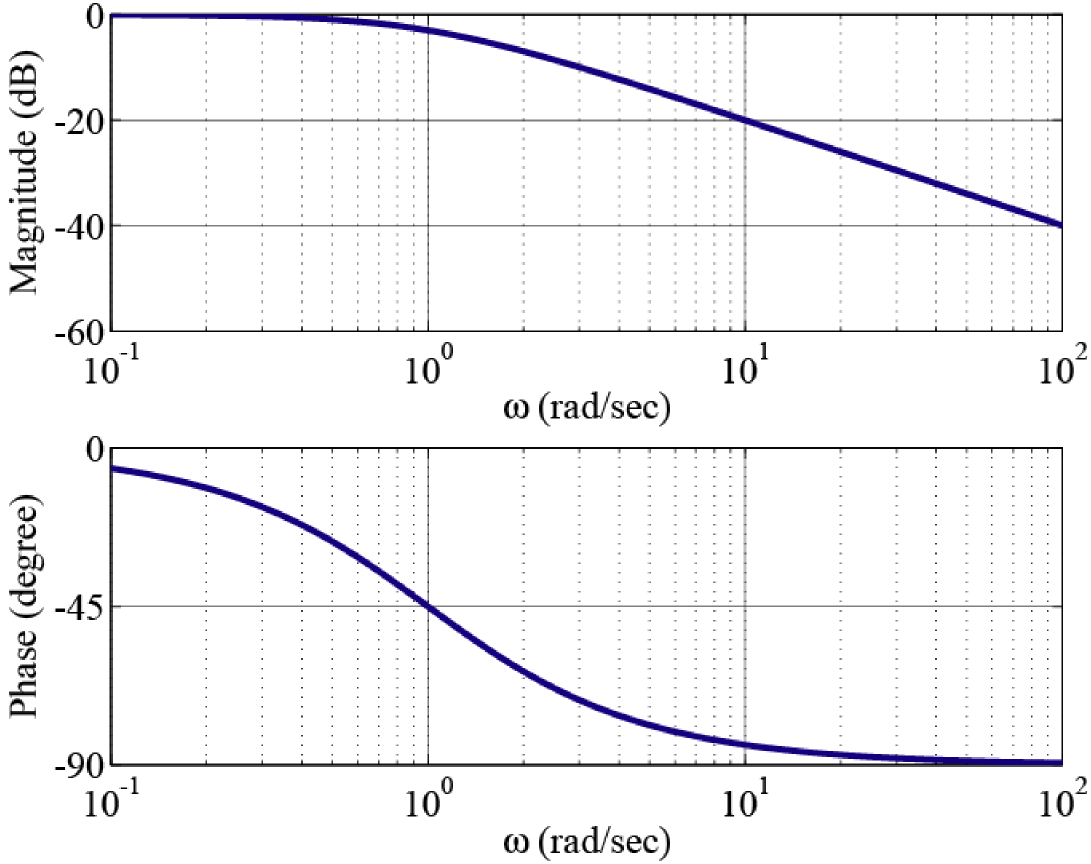
$\zeta = 0.7$



$\zeta = 0.9$

Bode Plot

$$\left|G(j\omega)\right|_{dB} = 20\log_{10}\left|G(j\omega)\right|$$



Bode Plot

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots}{s^m(s-p_1)(s-p_2)\cdots}$$

$$G(j\omega) = G(s)\big|_{s=j\omega} = K\frac{(j\omega-z_1)(j\omega-z_2)\cdots}{(j\omega)^m(j\omega-p_1)(j\omega-p_2)\cdots}$$

$$G(j\omega) = K_0\frac{(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots}{(j\omega)^m(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}$$

Bode Plot

$$\left|G(j\omega)\right|_{dB} = 20\log_{10}\left|G(j\omega)\right| = 20\log_{10}\left|K_0 \frac{(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots}{(j\omega)^m(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}\right|$$

$$= 20\log_{10}\left|K_0\right| \frac{\left|j\omega\tau_a+1\right|\left|j\omega\tau_b+1\right|\cdots}{\left|(j\omega)^m\right|\left|j\omega\tau_1+1\right|\left|j\omega\tau_2+1\right|\cdots}$$

$$= 20\log_{10}\left|K_0\right| + 20\log_{10}\left|j\omega\tau_a+1\right| + 20\log_{10}\left|j\omega\tau_b+1\right| + \cdots$$

$$-20\log_{10}\left|(j\omega)^m\right| - 20\log_{10}\left|j\omega\tau_1+1\right| - 20\log_{10}\left|j\omega\tau_2+1\right| - \cdots$$

Bode Plot

$$\angle G(j\omega) = \angle\left(K_0 \frac{(j\omega\tau_a + 1)(j\omega\tau_b + 1)\cdots}{(j\omega)^m (j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\cdots} \right)$$

$$= \angle K_0 + \angle(j\omega\tau_a + 1) + \angle(j\omega\tau_b + 1) + \cdots$$

$$- \angle(j\omega)^m - \angle(j\omega\tau_1 + 1) - \angle(j\omega\tau_2 + 1) - \cdots$$

Bode Plot

$$K_0 s^m$$

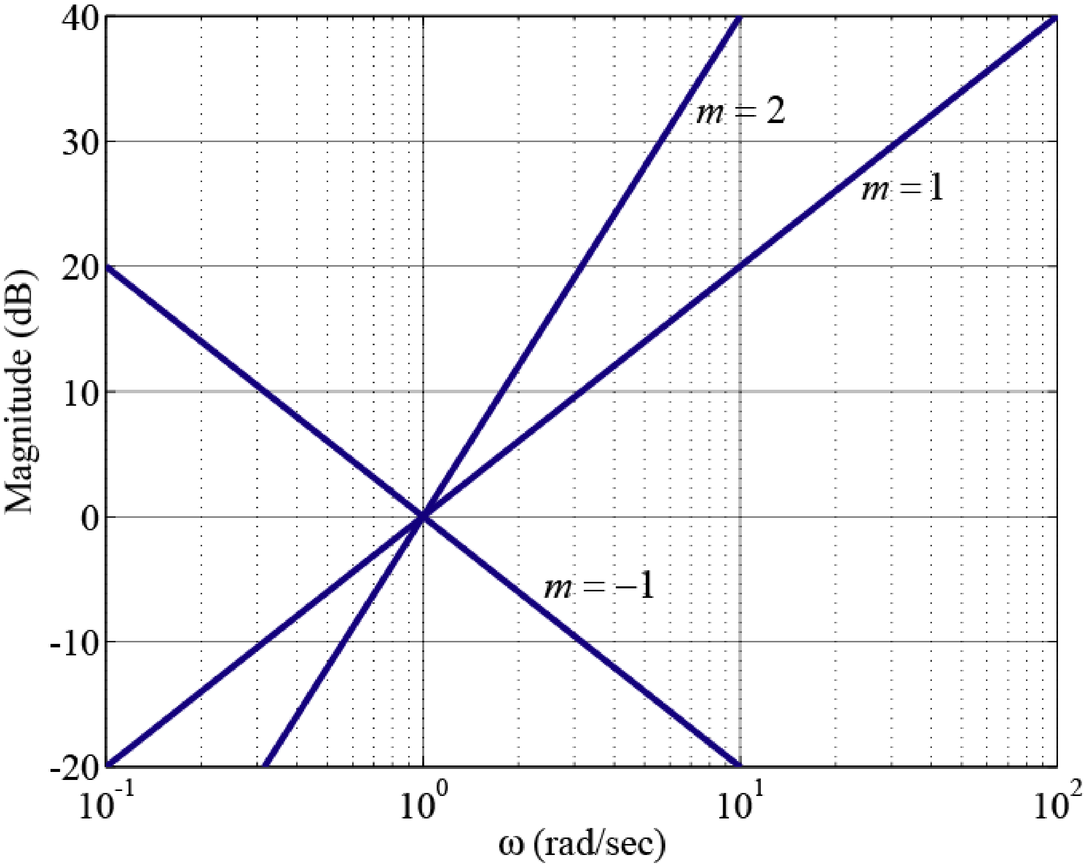
$$(s\tau+1),1/(s\tau+1)$$

$$\left[(s/\omega_n)^2+(2\zeta s/\omega_n)+1\right],1/\left[(s/\omega_n)^2+(2\zeta s/\omega_n)+1\right]$$

$K_0 s^m$

$$20\log_{10}\left|K_0\left(j\omega\right)^m\right| = 20\log_{10}\left|K_0\right| + 20\log_{10}\left|\left(j\omega\right)^m\right|$$

$$= 20\log_{10}\left|K_0\right| + 20m\log_{10}\left|\omega\right|$$



$K_0 s^m$

- 위상

$$\angle\left(K_0 (j\omega)^m\right) = \angle K_0 + \angle (j\omega)^m$$

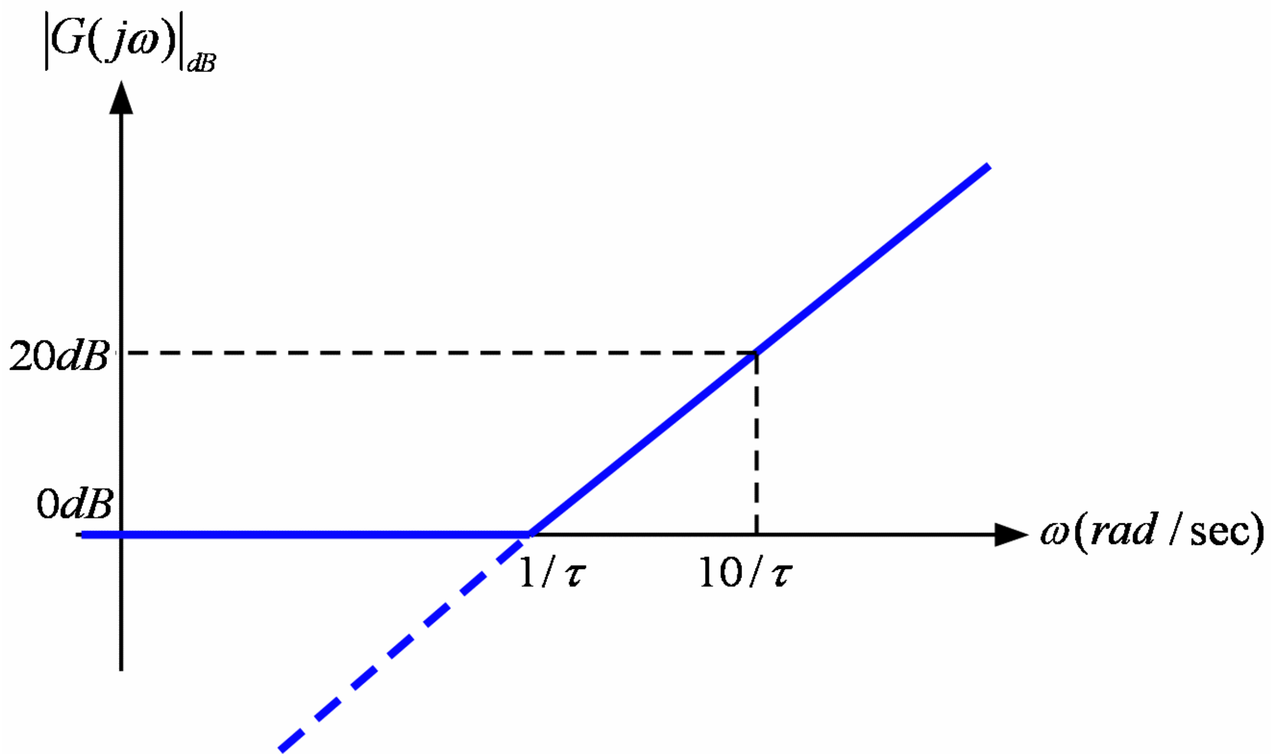
$$= \angle K_0 + m\angle (j\omega) = \angle K_0 + m \times 90°$$

$$\left(s\tau+1\right)$$

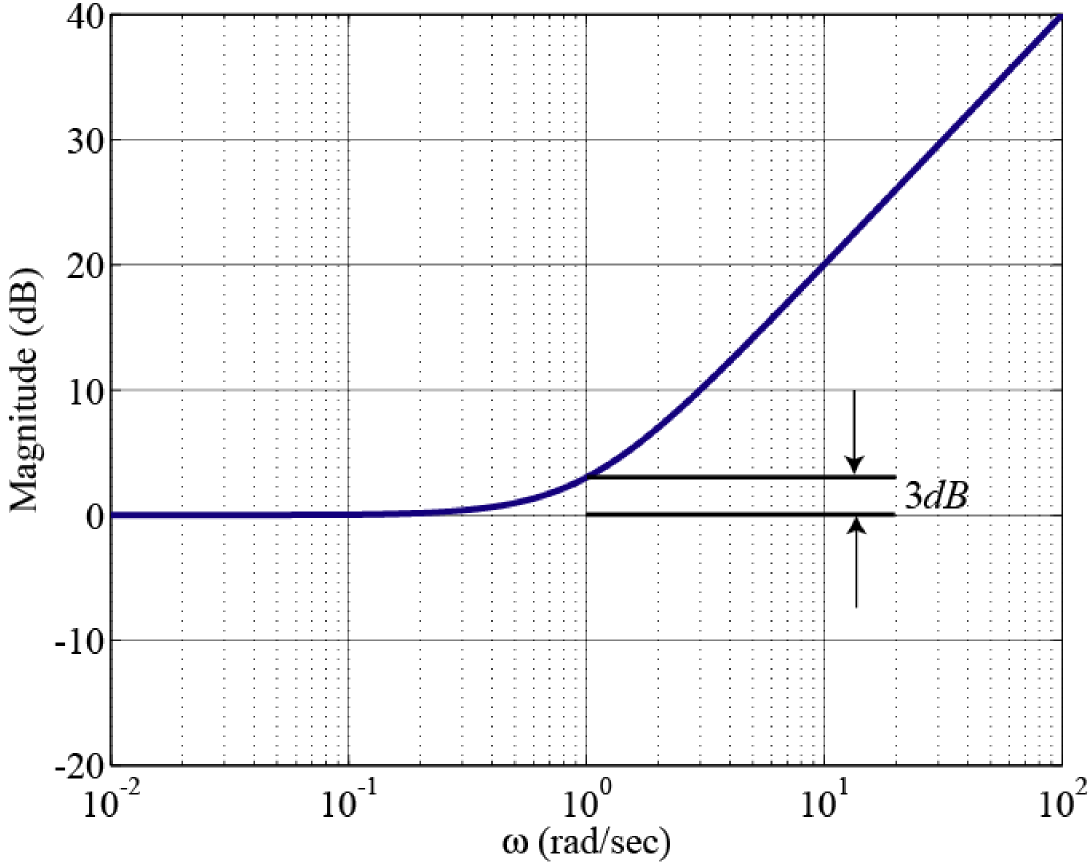
$$\left|j\omega\tau+1\right|_{dB}=20\log_{10}\left|j\omega\tau+1\right|\approx 20\log_{10}1=0 \qquad \omega \ll 1/\tau$$

$$\left|j\omega\tau+1\right|_{dB}=20\log_{10}\left|j\omega\tau+1\right|\approx 20\log_{10}\left|\omega\tau\right| \qquad \omega \gg 1/\tau$$

$$\left|j\omega\tau+1\right|_{dB}=20\log_{10}\left|j\omega\tau+1\right|=20\log_{10}\sqrt{\left(\omega\tau\right)^2+1}$$



$$\left(s\tau + 1\right)$$

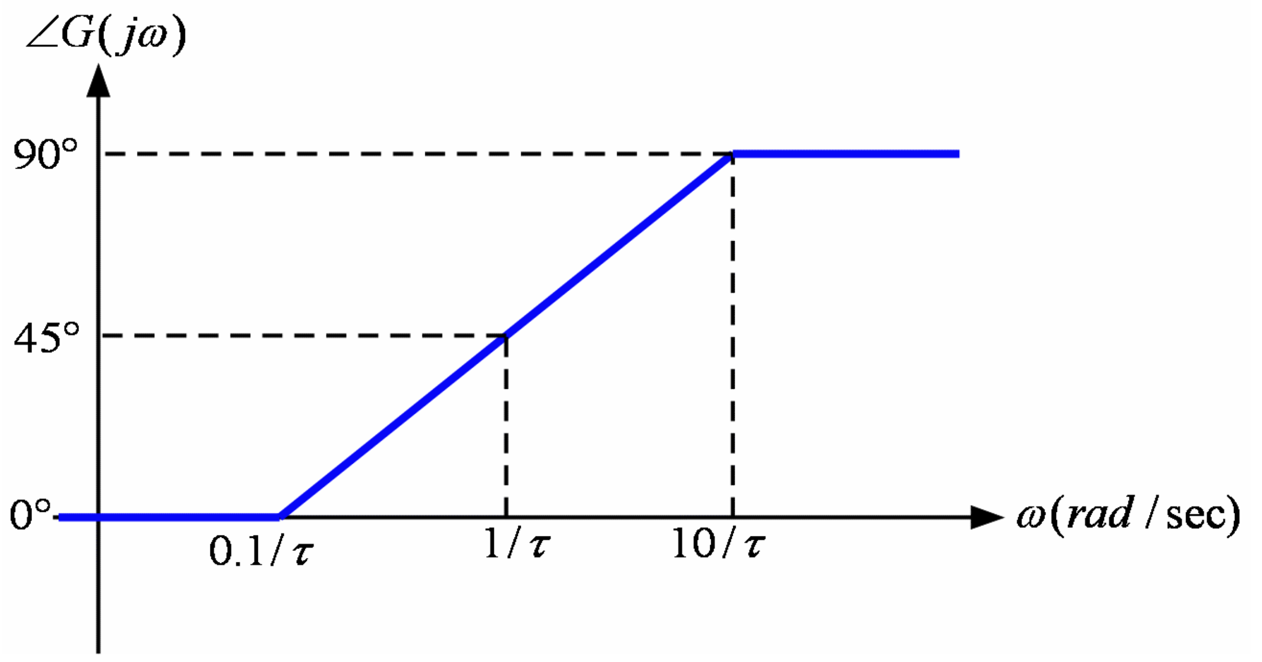


$$\left(s\tau+1\right)$$

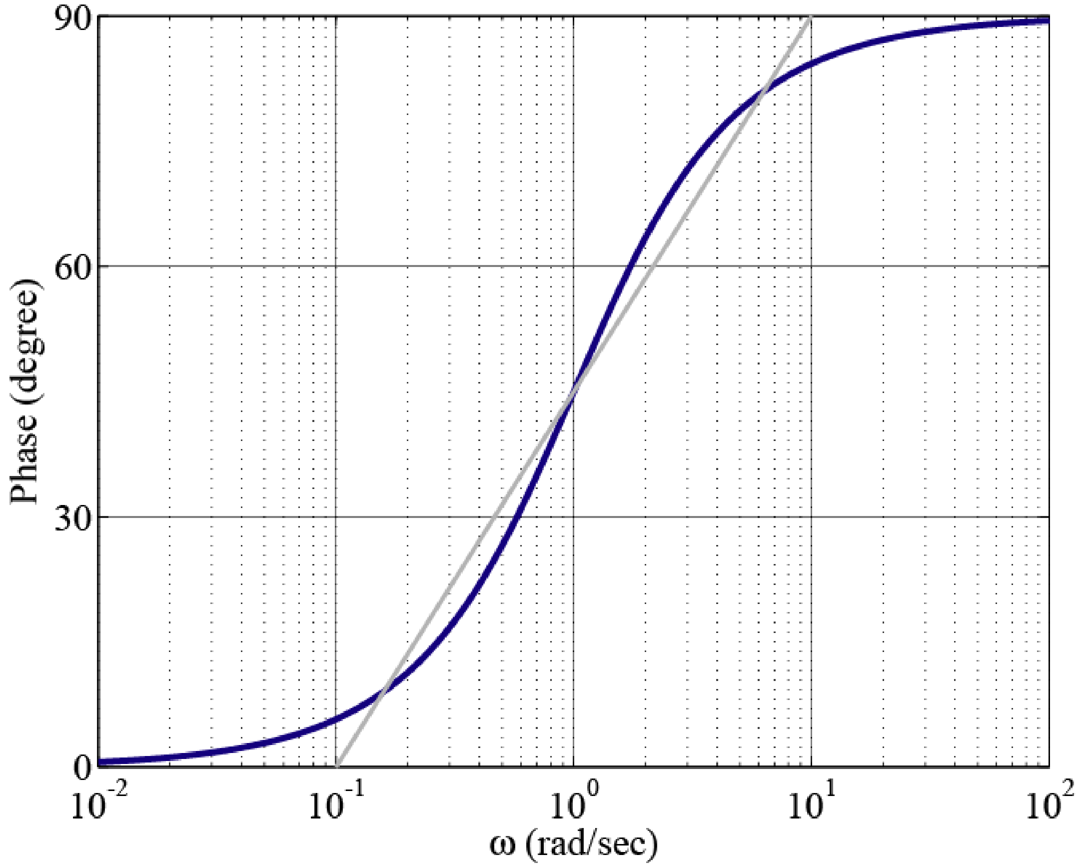
$$\angle\left(j\omega\tau+1\right)\approx\angle 1=0^\circ \qquad \omega\ll 1/\tau$$

$$\angle\left(j\omega\tau+1\right)=\angle\left(j+1\right)=45^\circ \qquad \omega=1/\tau$$

$$\angle\left(j\omega\tau+1\right)\approx\angle\left(j\omega\tau\right)=90^\circ \qquad \omega\gg 1/\tau$$



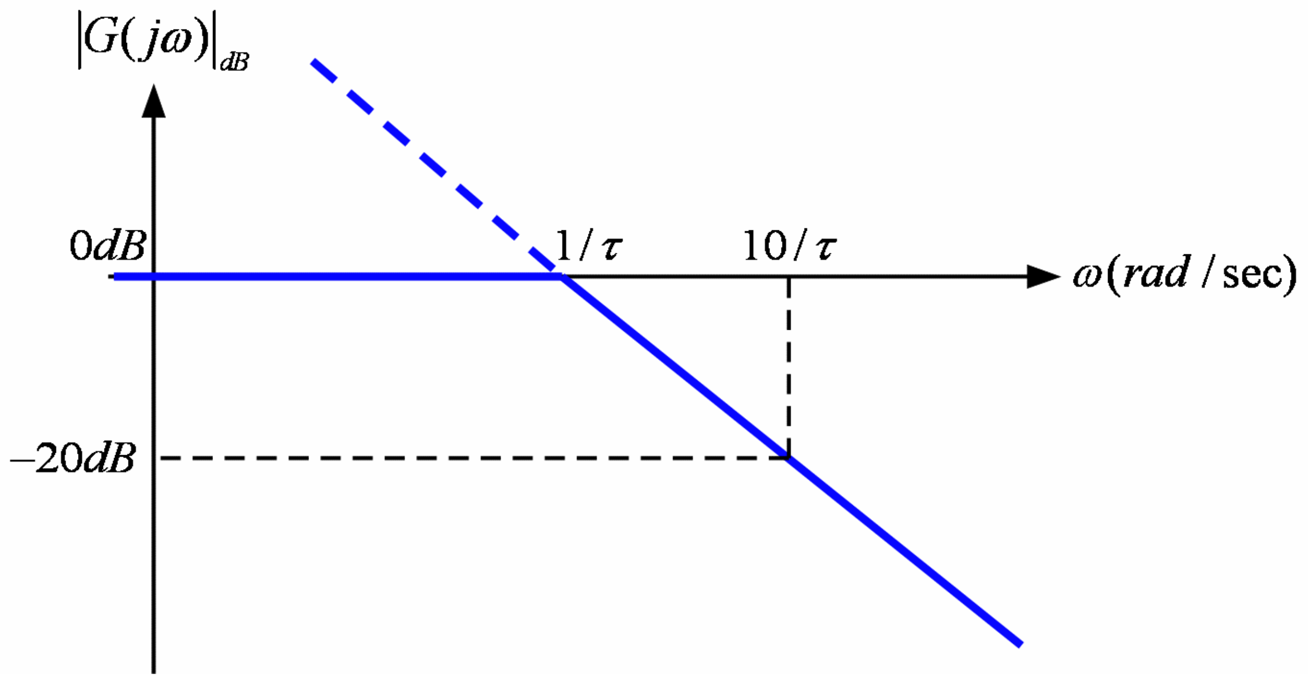
$$(s\tau + 1)$$



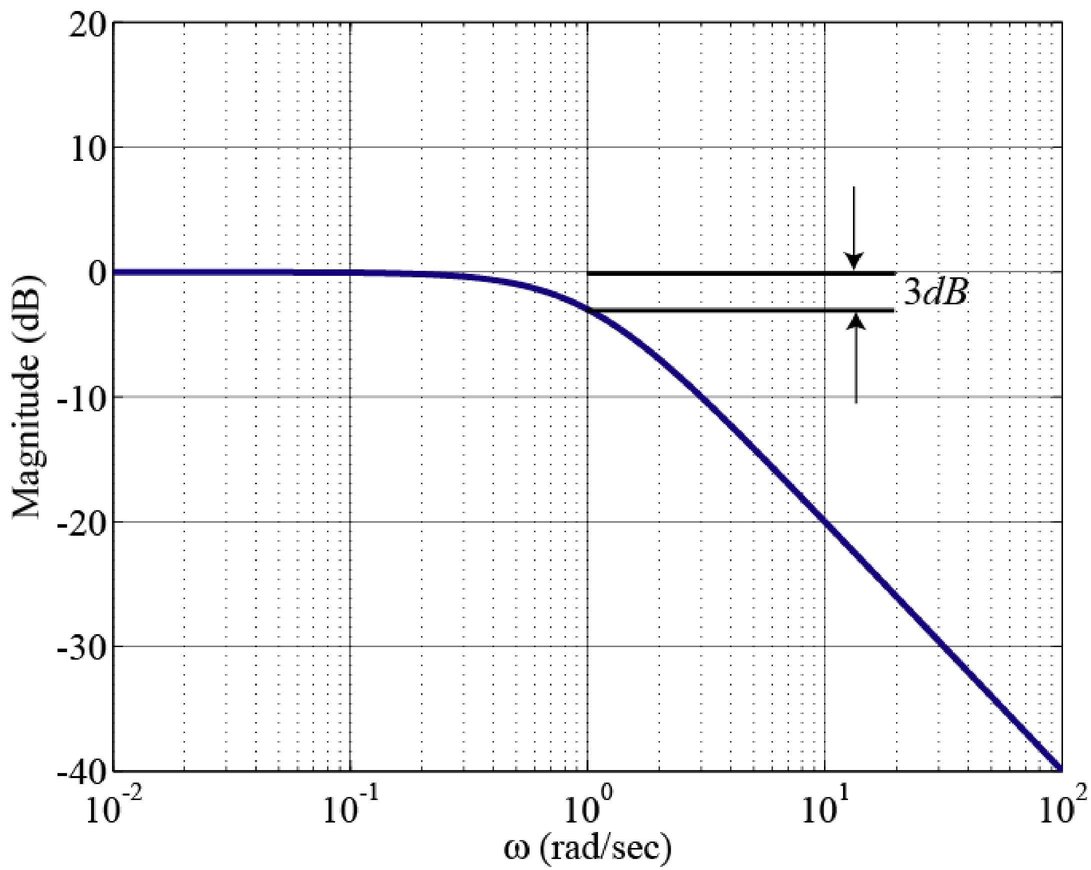
$1/(s\tau+1)$

$$\left|\frac{1}{j\omega\tau+1}\right|_{dB} = -20\log_{10}\left|j\omega\tau+1\right| \approx -20\log_{10} 1 = 0 \quad \omega \ll 1/\tau$$

$$\left|\frac{1}{j\omega\tau+1}\right|_{dB} = -20\log_{10}\left|j\omega\tau+1\right| \approx -20\log_{10}\left|\omega\tau\right| \quad \omega \gg 1/\tau$$



$1/\left(s\tau+1\right)$

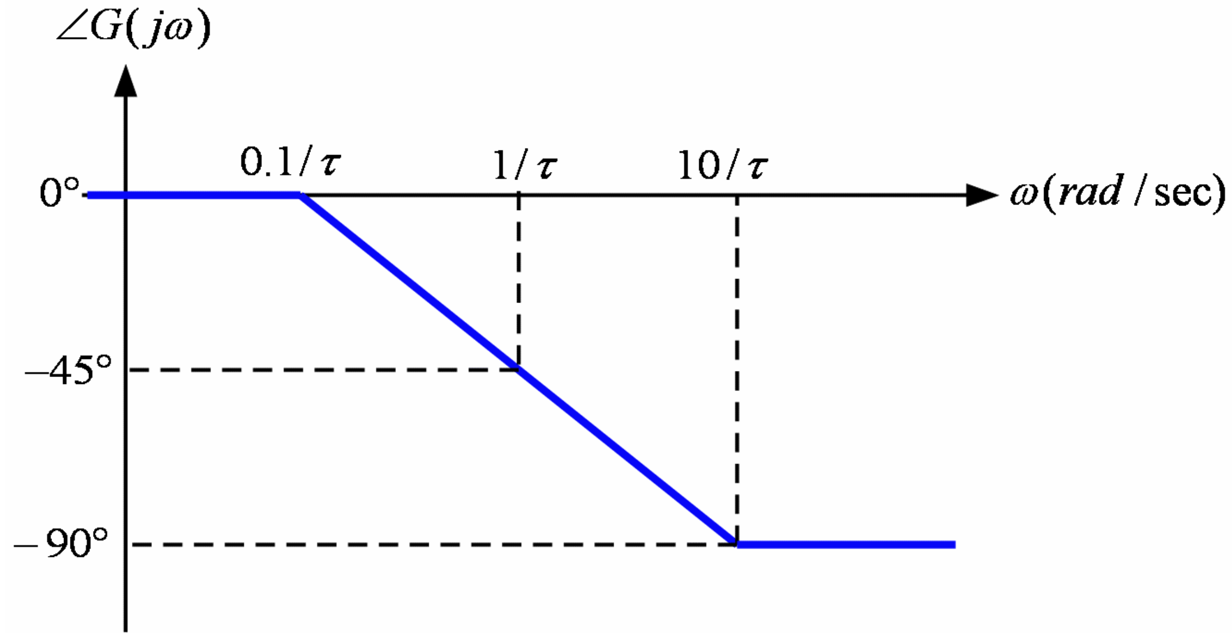


$1/\left(s\tau+1\right)$

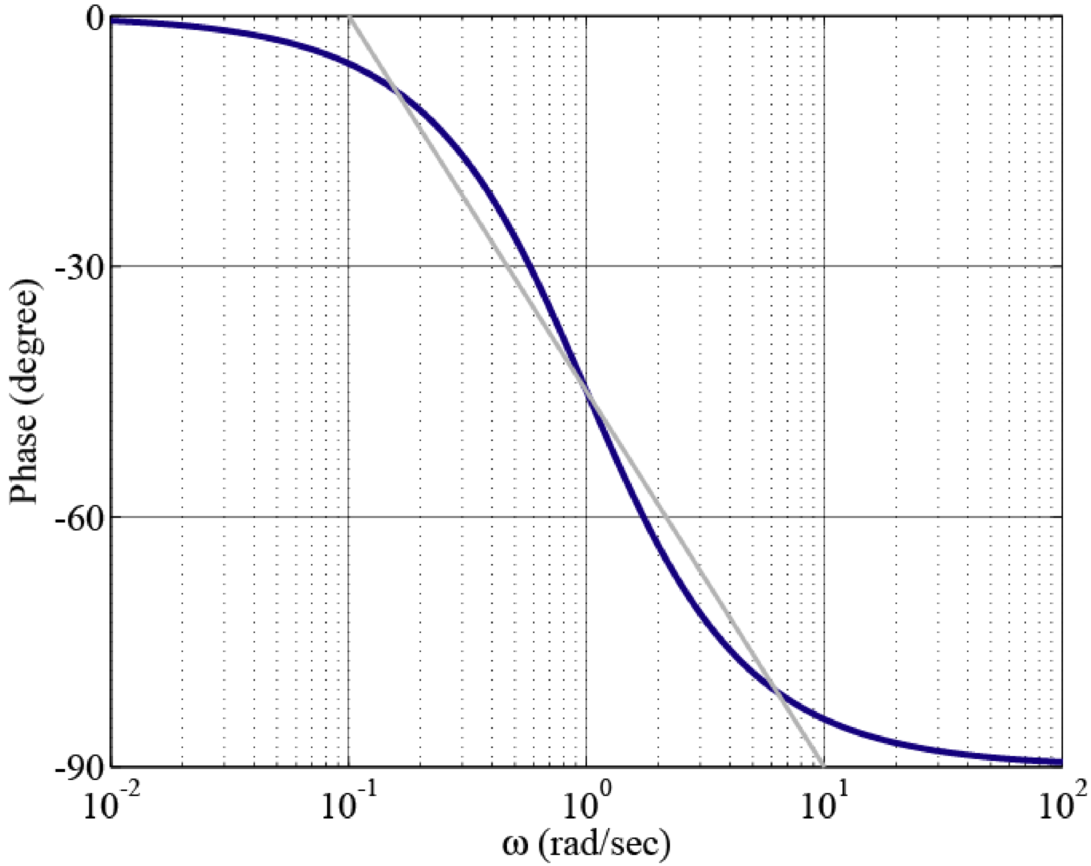
$$\angle\left(\frac{1}{j\omega\tau+1}\right)\approx-\angle 1=0^\circ \qquad \omega\ll 1/\tau$$

$$\angle\left(\frac{1}{j\omega\tau+1}\right)=-\angle\left(j+1\right)=-45^\circ \qquad \omega=1/\tau$$

$$\angle\left(\frac{1}{j\omega\tau+1}\right)\approx-\angle\left(j\omega\tau\right)=-90^\circ \qquad \omega\gg 1/\tau$$



$1/(s\tau+1)$



Example

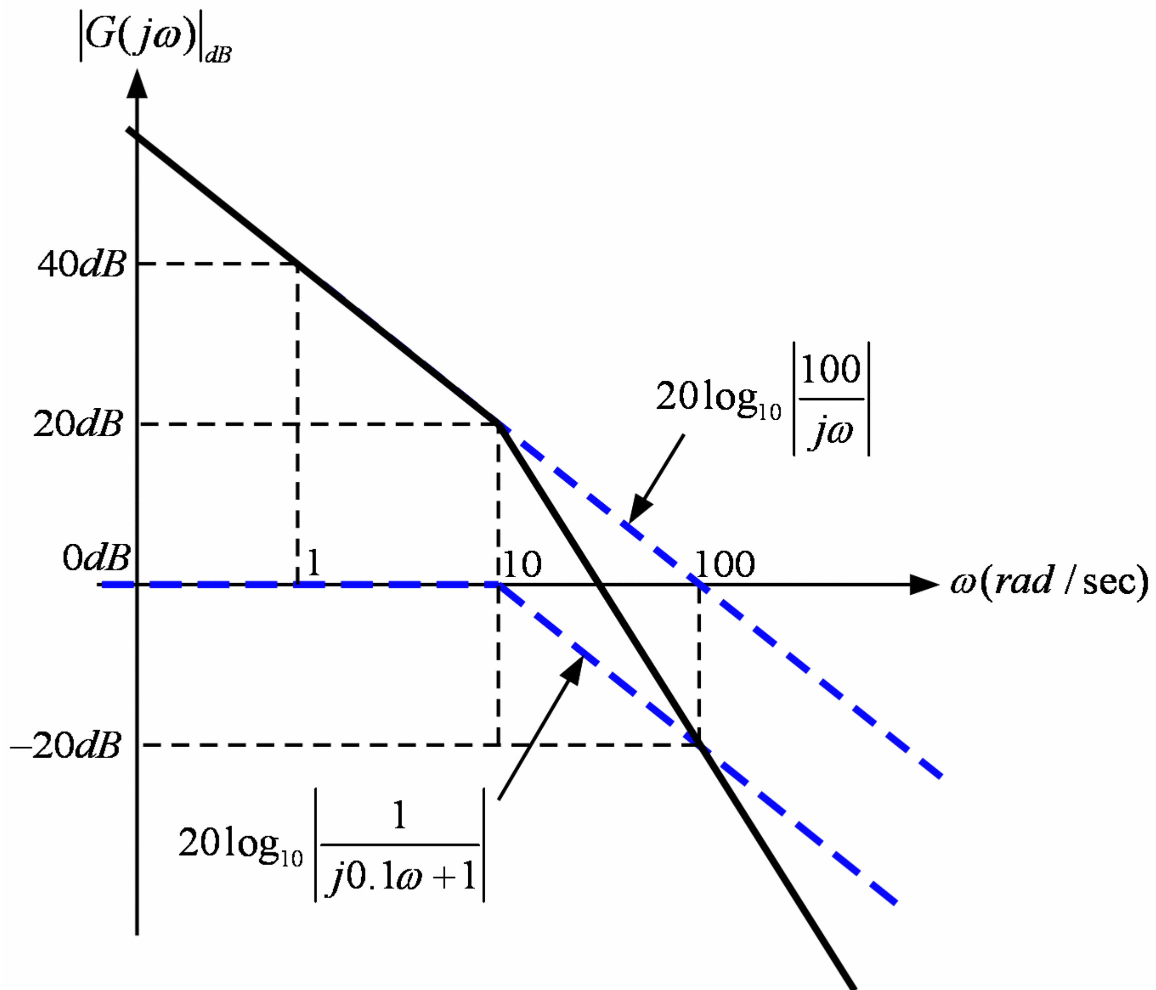
$$G(s)=\frac{1000}{s\left(s+10\right)}$$

$$G(j\omega)=\left.\frac{1000}{s\left(s+10\right)}\right|_{s=j\omega}=\frac{100}{j\omega\left(j0.1\omega+1\right)}$$

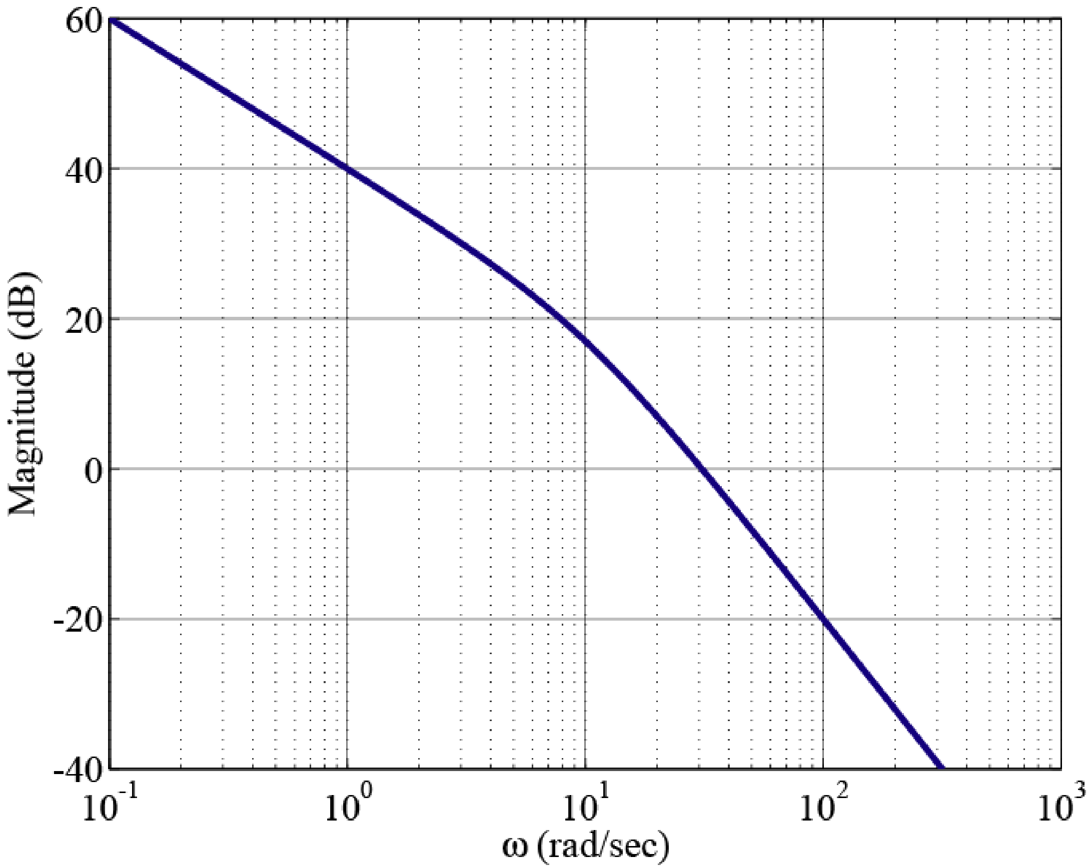
$$20\log_{10}\left|\frac{100}{j\omega}\right|$$

$$=20\log_{10}100-20\log\left|\omega\right|$$

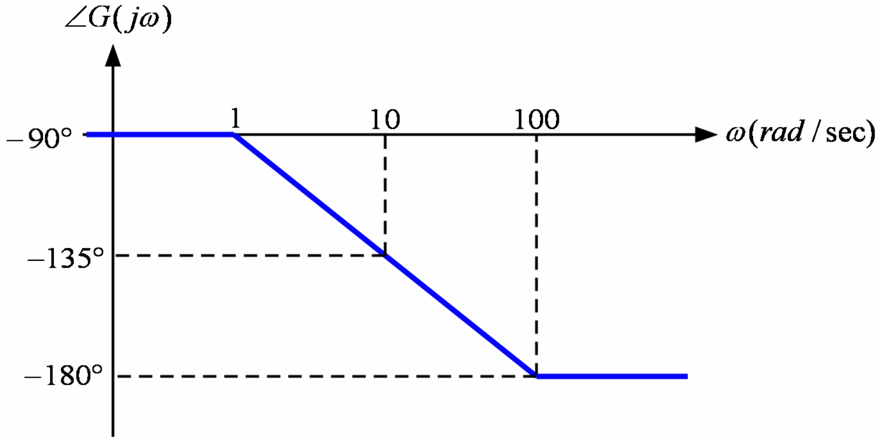
$$=40-20\log\left|\omega\right|$$



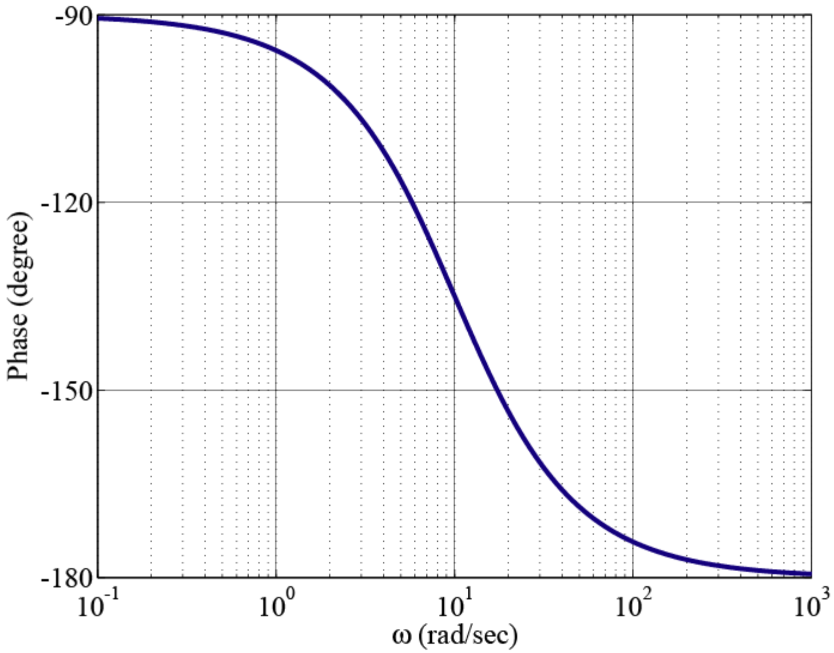
Example



Example



∠G(jω)
−90°
1
10
100
ω(rad / sec)
−135°
−180°

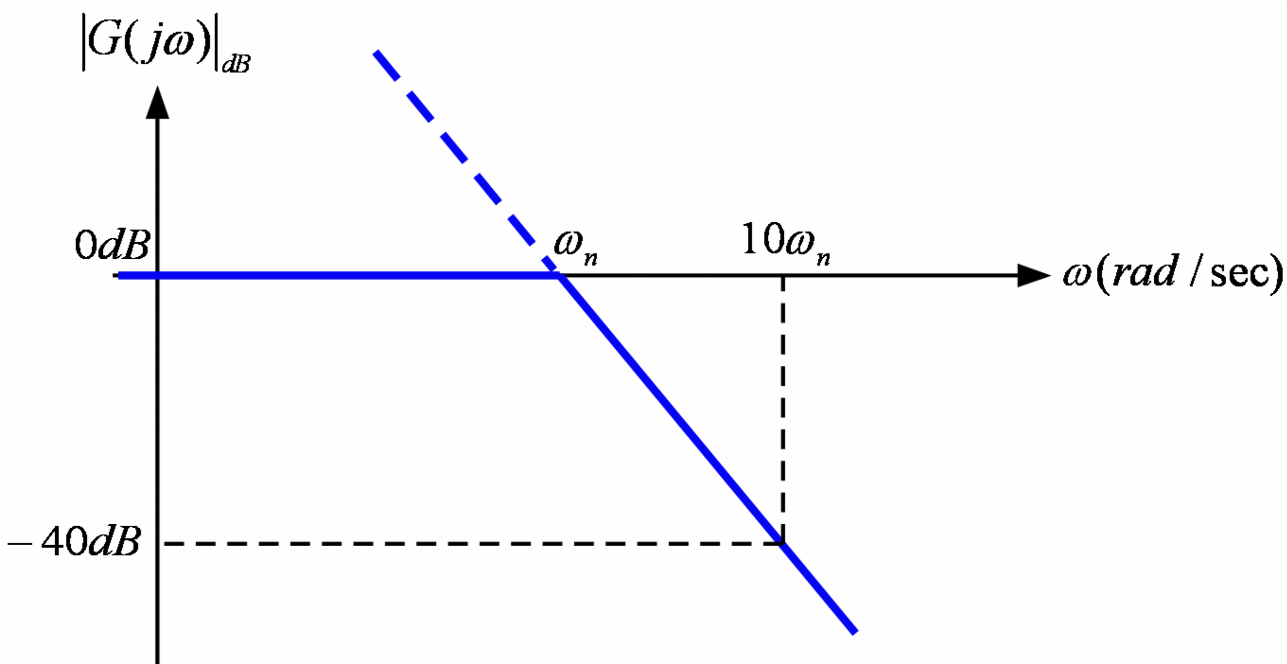


-90
-120
-150
-180
Phase (degree)
10^-1
10^0
10^1
10^2
10^3
ω (rad/sec)

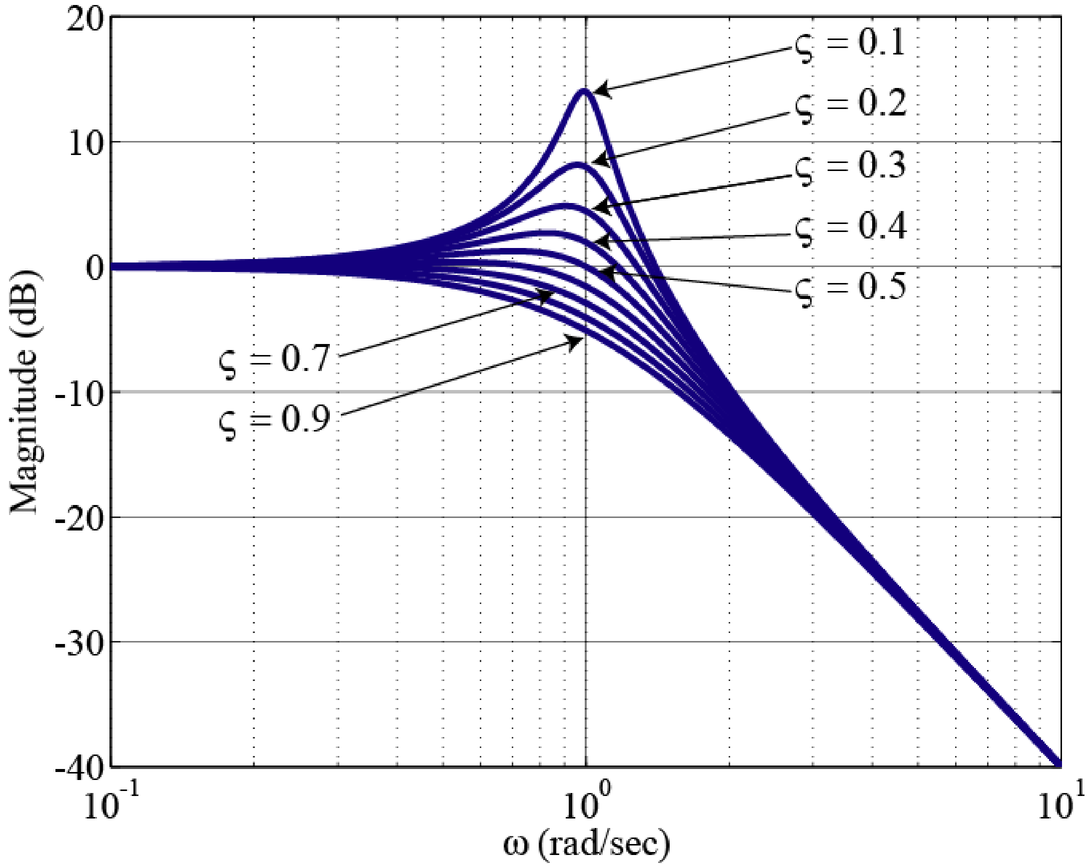
$$1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$$

$$\left|\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\varsigma j\omega/\omega_n\right)+1}\right|_{dB}\approx -20\log_{10}1=0 \qquad \omega\ll\omega_n$$

$$\left|\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\varsigma j\omega/\omega_n\right)+1}\right|_{dB}\approx -20\log_{10}\left|\left(\omega/\omega_n\right)^2\right| \qquad \omega\gg\omega_n$$



$$1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$$



$$1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$$

$$\left|\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\zeta j\omega/\omega_n\right)+1}\right|_{\omega=\omega_n}=\frac{1}{2\zeta}$$

$$20\log\frac{1}{2\zeta}=-20\log 2\zeta$$

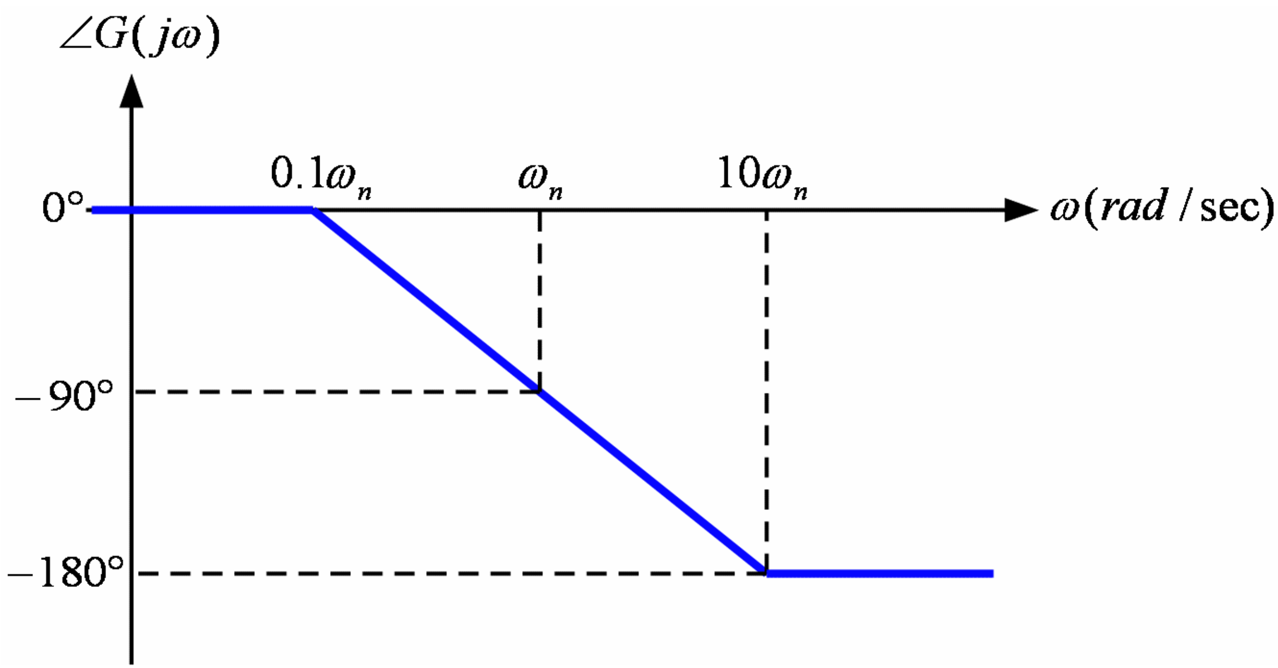
$$1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$$

$$\angle\left(\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\zeta j\omega/\omega_n\right)+1}\right)\approx -\angle 1 = 0^\circ \qquad \omega \ll \omega_n$$

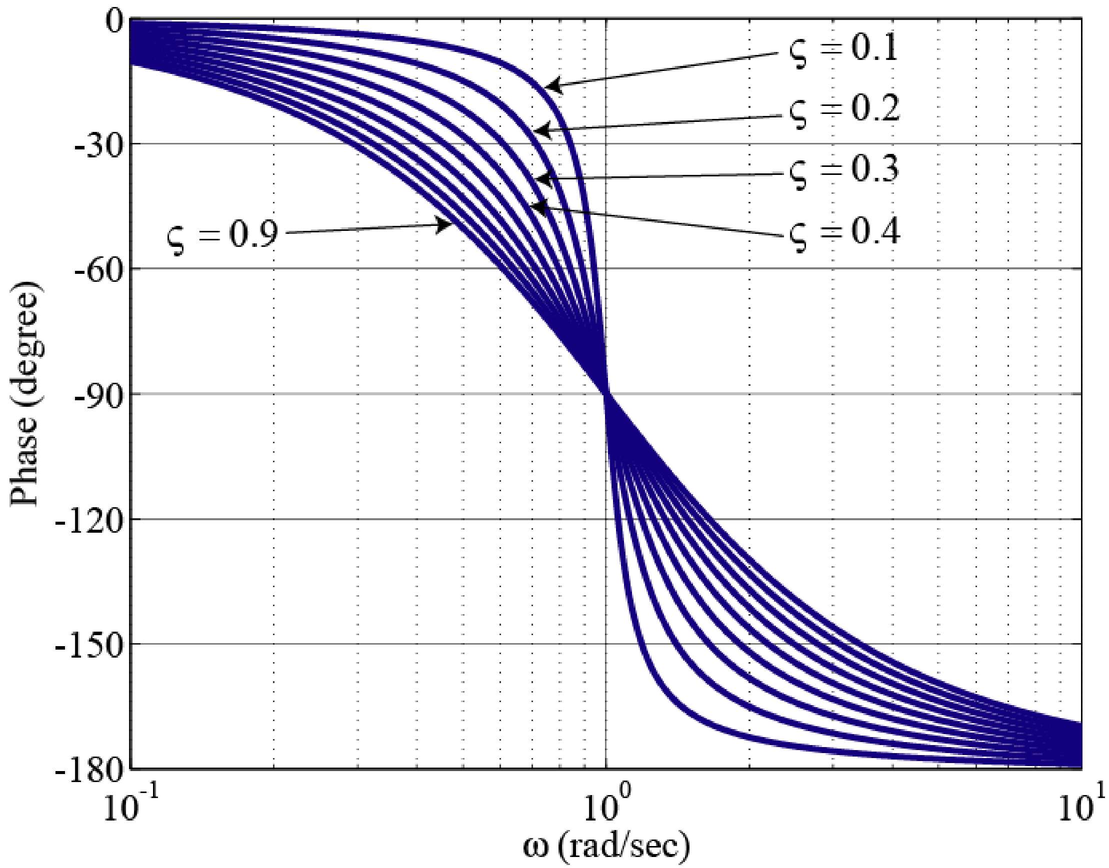
$$\angle\left(\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\zeta j\omega/\omega_n\right)+1}\right)= -\angle\left(2\varsigma j\right) = -90^\circ \qquad \omega = \omega_n$$

$$\angle\left(\frac{1}{\left(j\omega/\omega_n\right)^2+\left(2\zeta j\omega/\omega_n\right)+1}\right)\approx -\angle\left(-\omega^2/\omega_n^2\right) = -180^\circ \qquad \omega \gg \omega_n$$

$$1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$$



$$1/\left[\left(s/\omega_n\right)^2+\left(2\zeta s/\omega_n\right)+1\right]$$



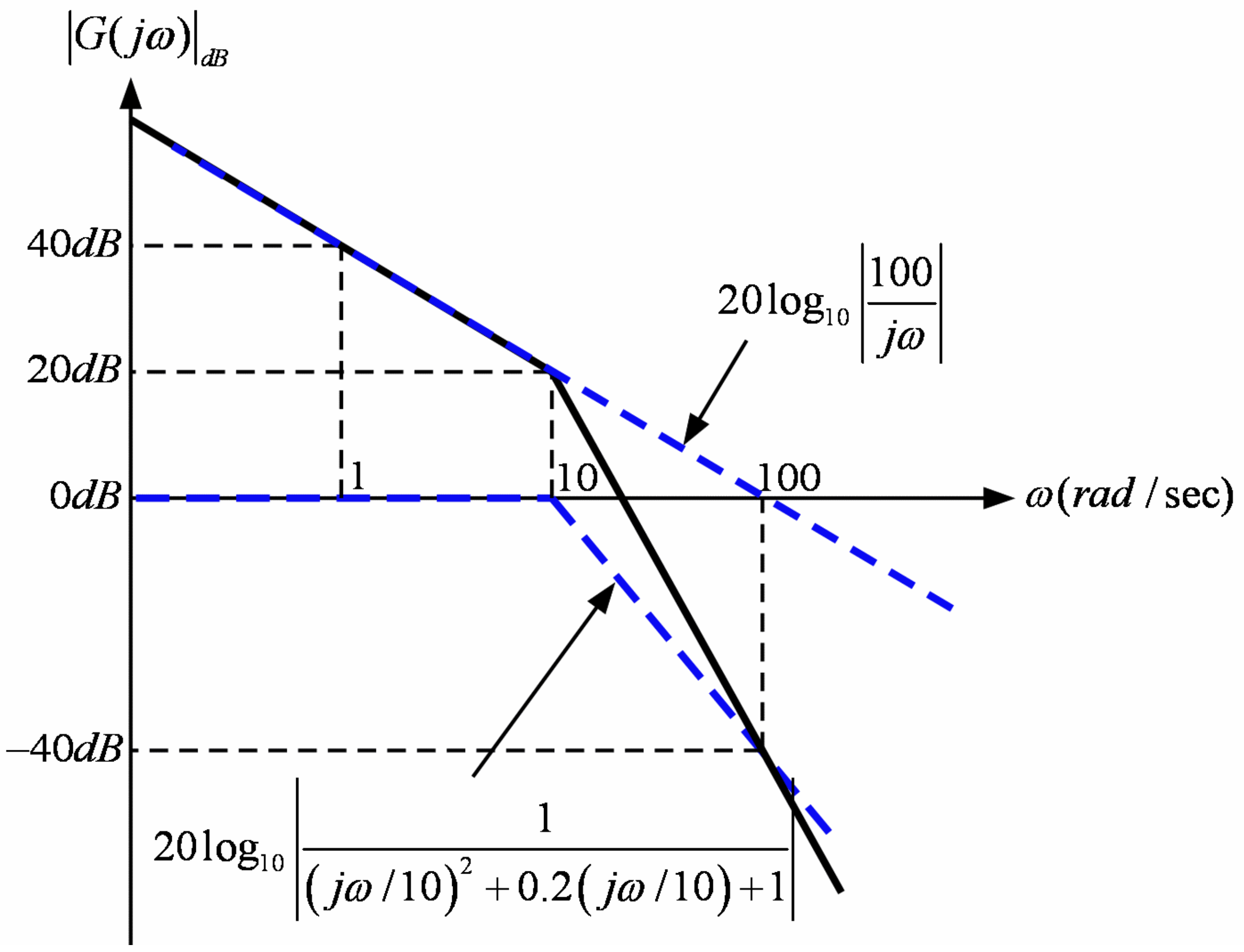
Example

$$G(s) = \frac{10000}{s\left(s^2 + 2s + 100\right)}$$

$$G(j\omega) = \left.\frac{100}{s\left(s^2/100 + 2s/100 + 1\right)}\right|_{s=j\omega}$$

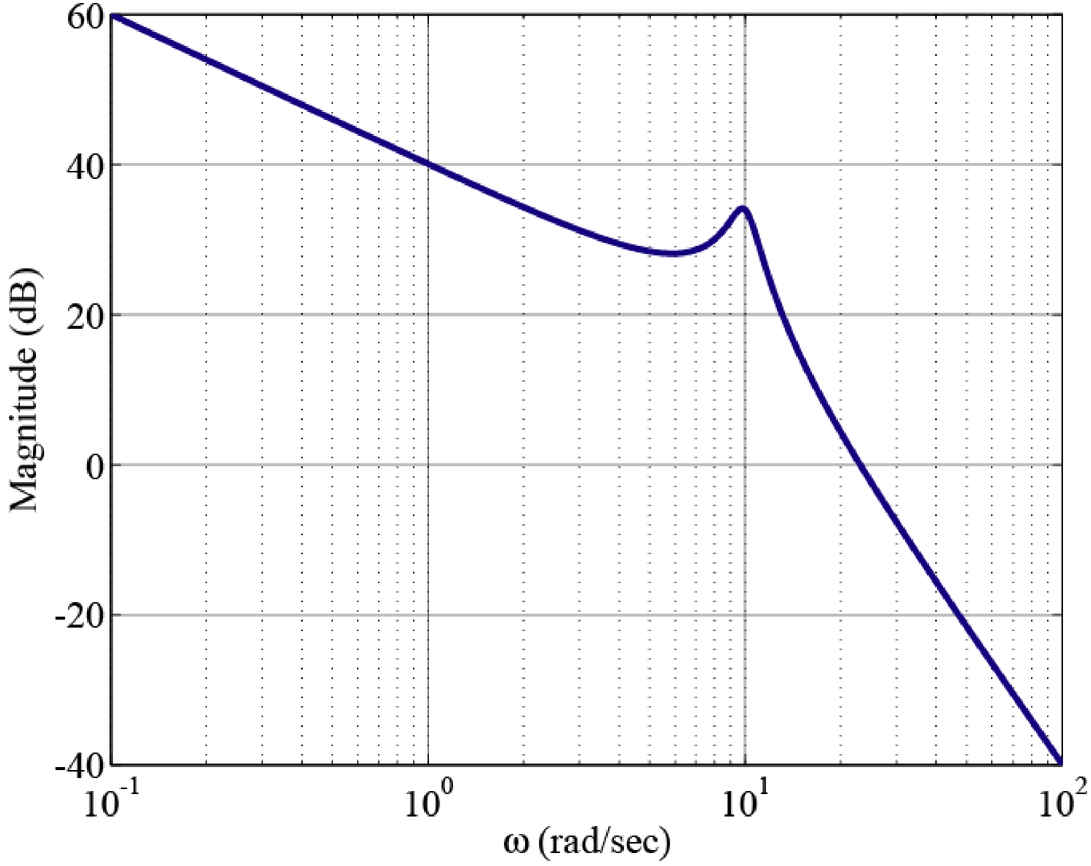
$$= \frac{100}{j\omega\left(\left(j\omega/10\right)^2 + 0.2\left(j\omega/10\right) + 1\right)}$$

Example



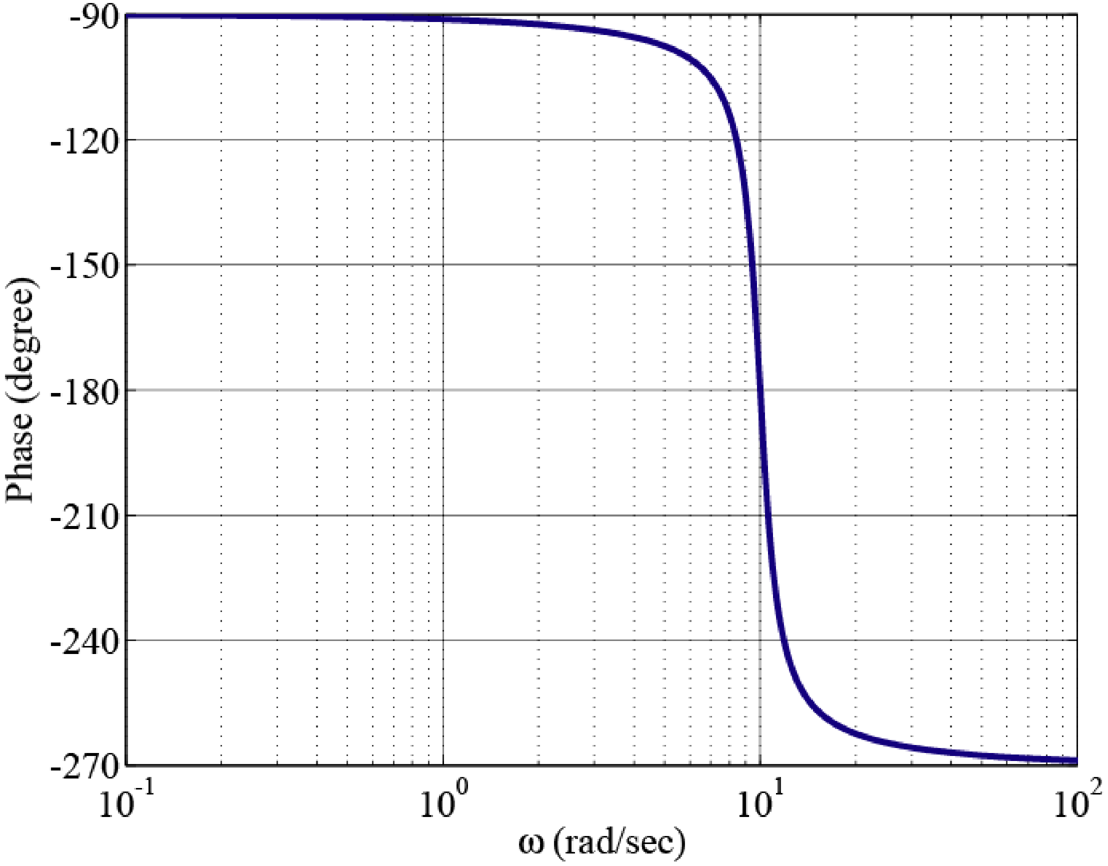
$|G(j\omega)|_{dB}$
40dB
20dB
0dB
-40dB
1
10
100
$\omega(rad/\sec)$
$20\log_{10}\left|\frac{100}{j\omega}\right|$
$20\log_{10}\left|\frac{1}{(j\omega/10)^2+0.2(j\omega/10)+1}\right|$

Example



$$20\log_{10}\frac{1}{2\zeta}=20\log_{10}5=14dB$$

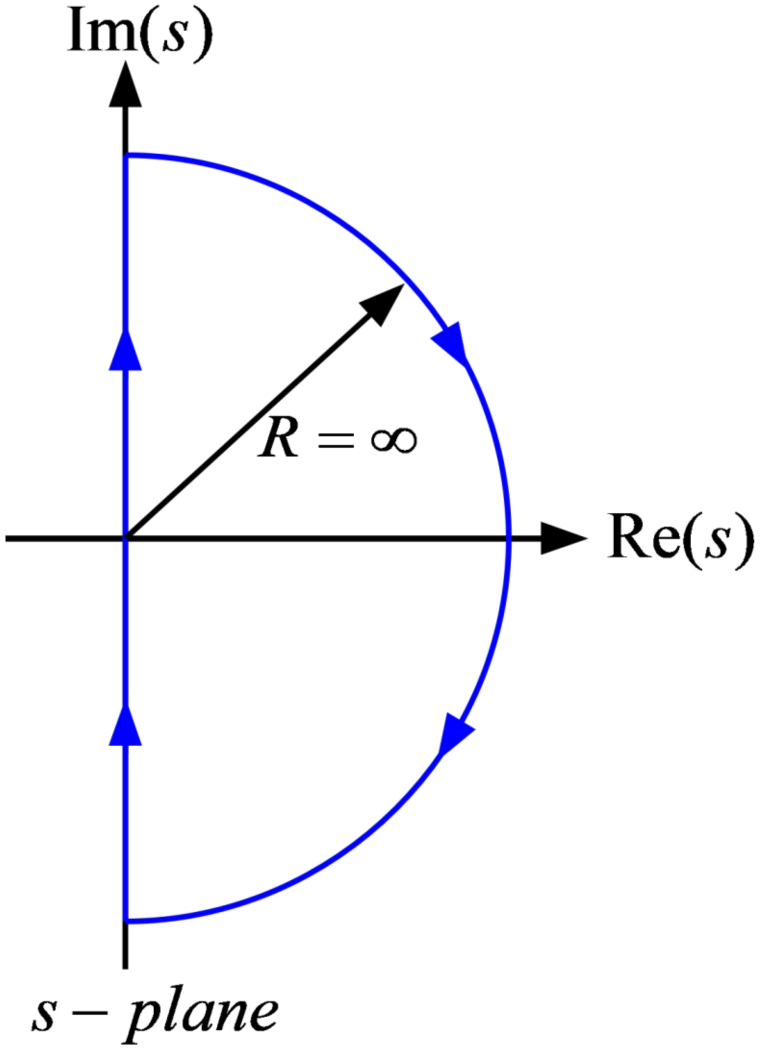
Example



-90
-120
-150
-180
-210
-240
-270
Phase (degree)
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)

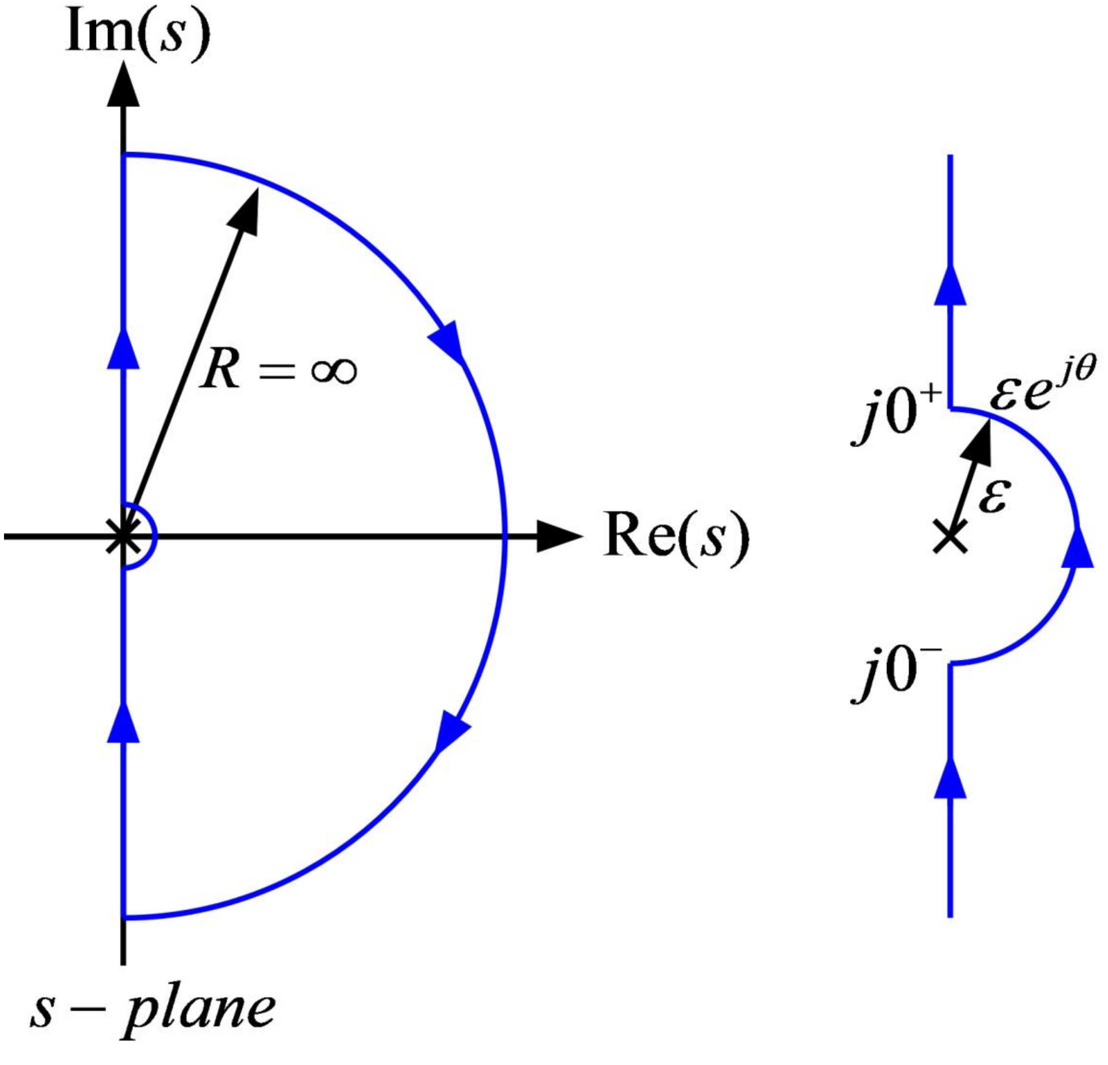
Nyquist Plot

- Nyquist path

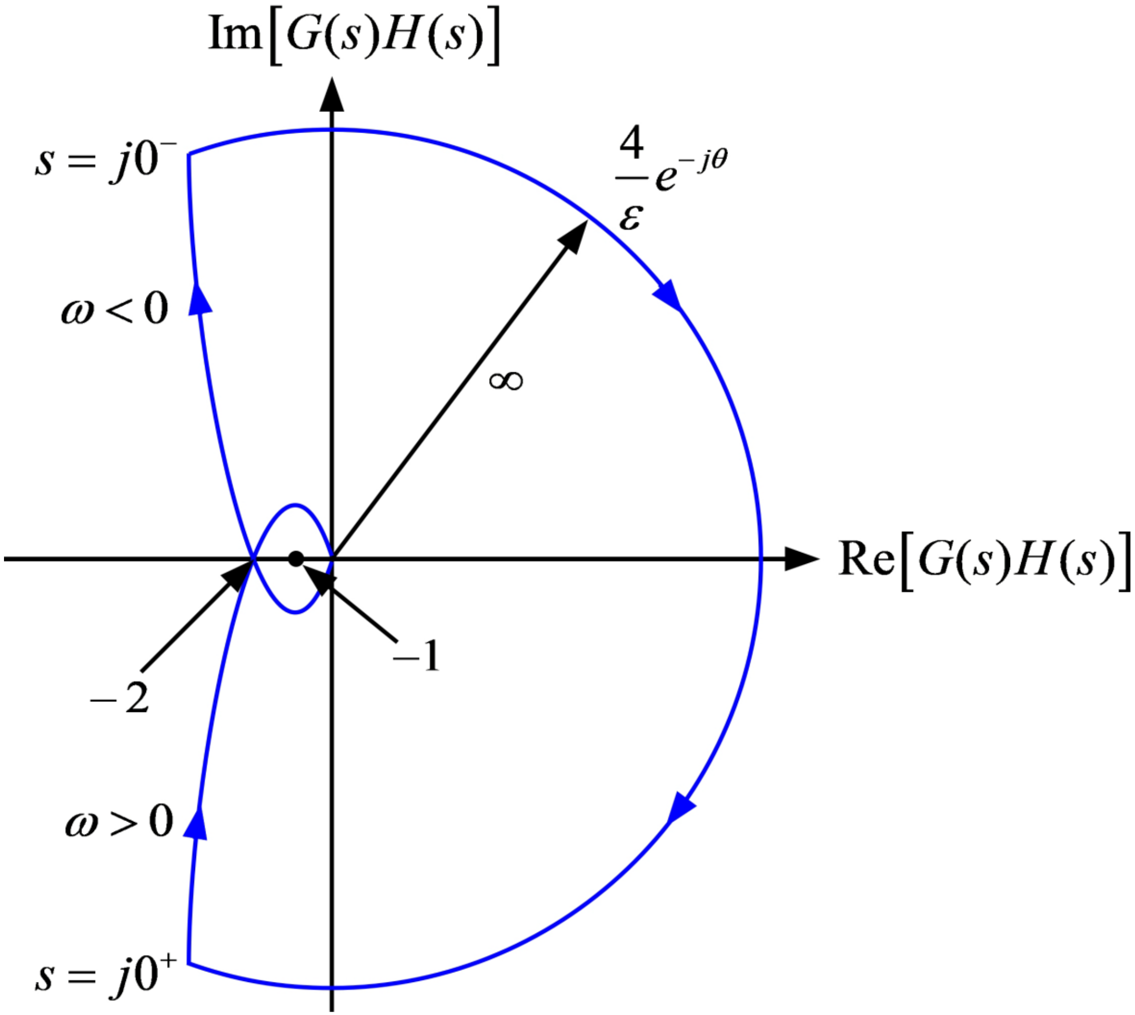


Nyquist Plot

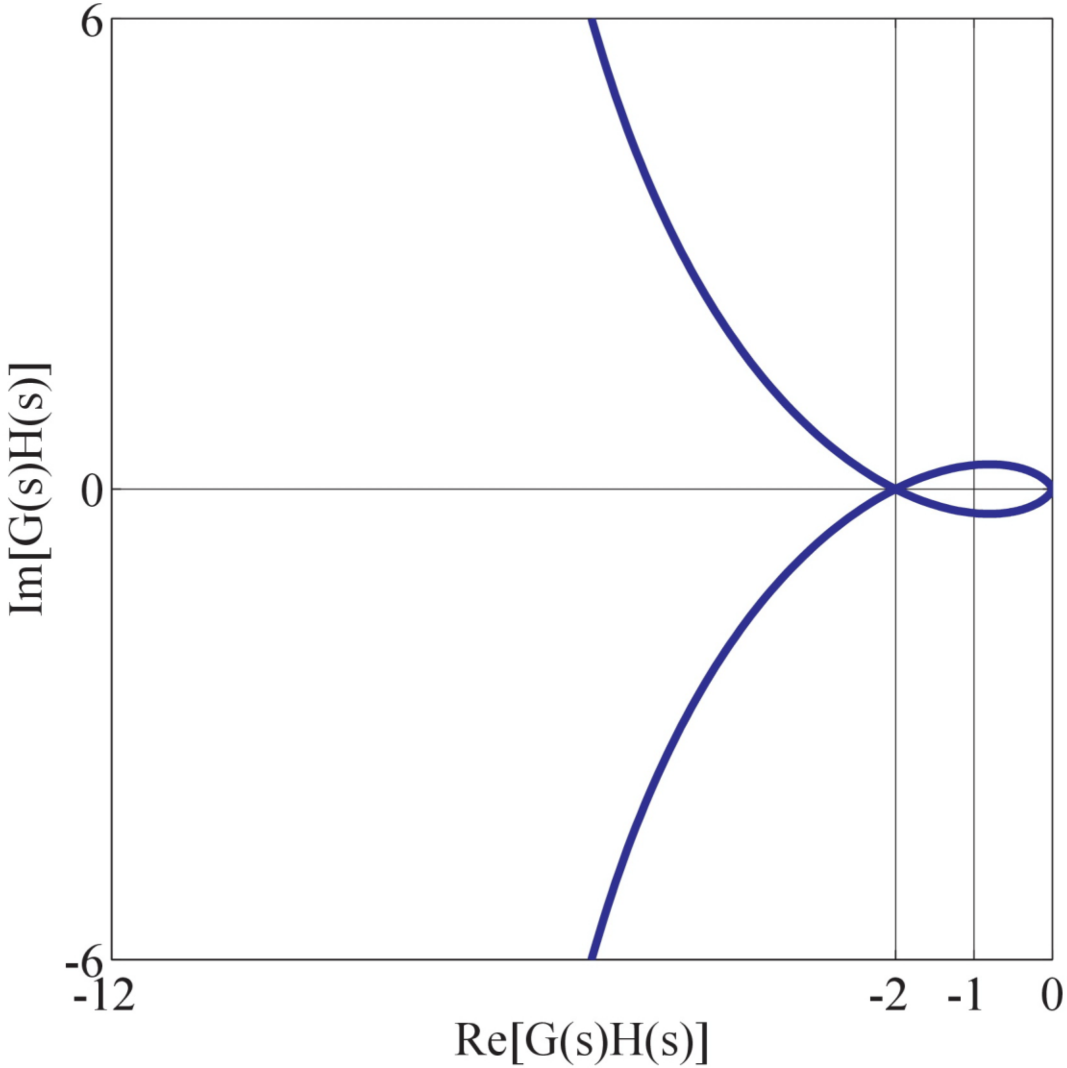
- Nyquist path



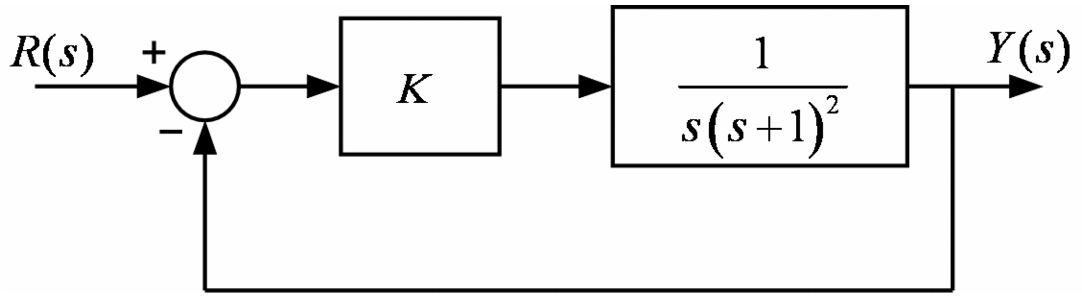
Nyquist Plot

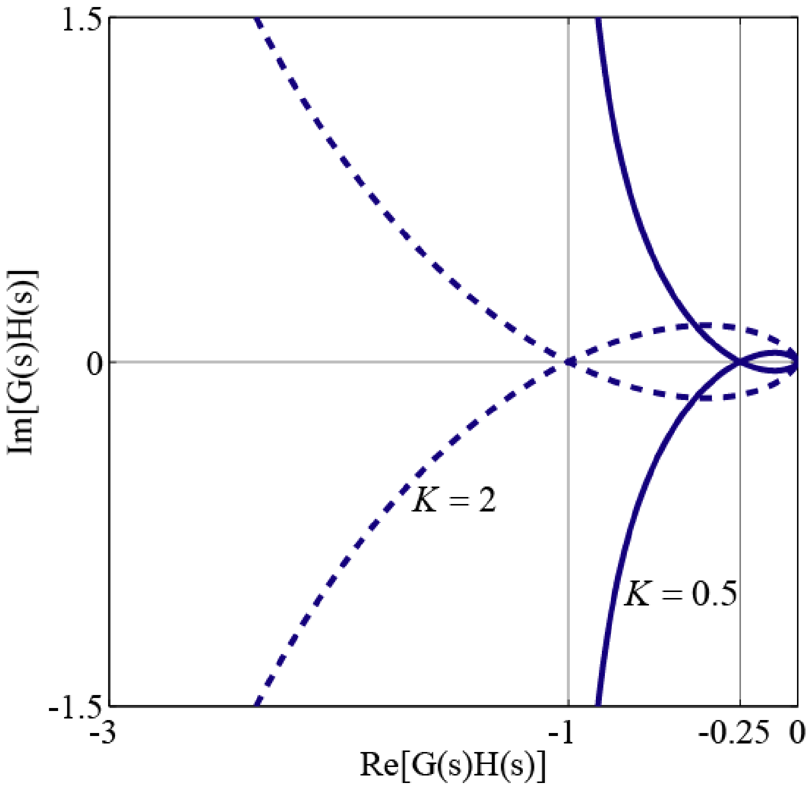


Nyquist Plot

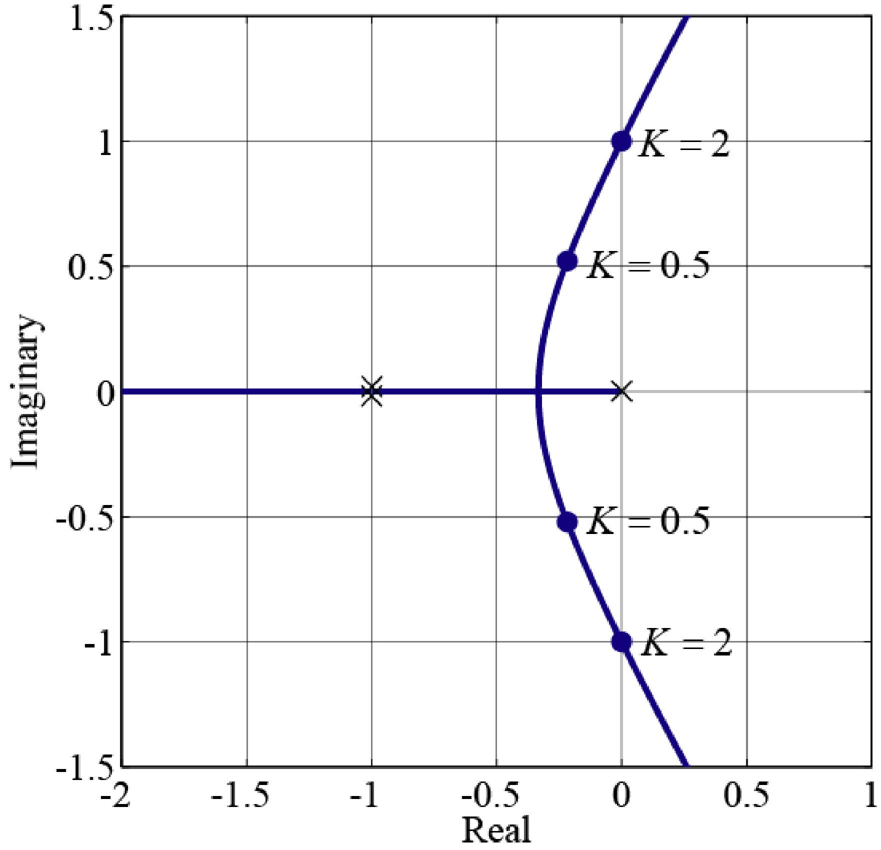


Stability Margin

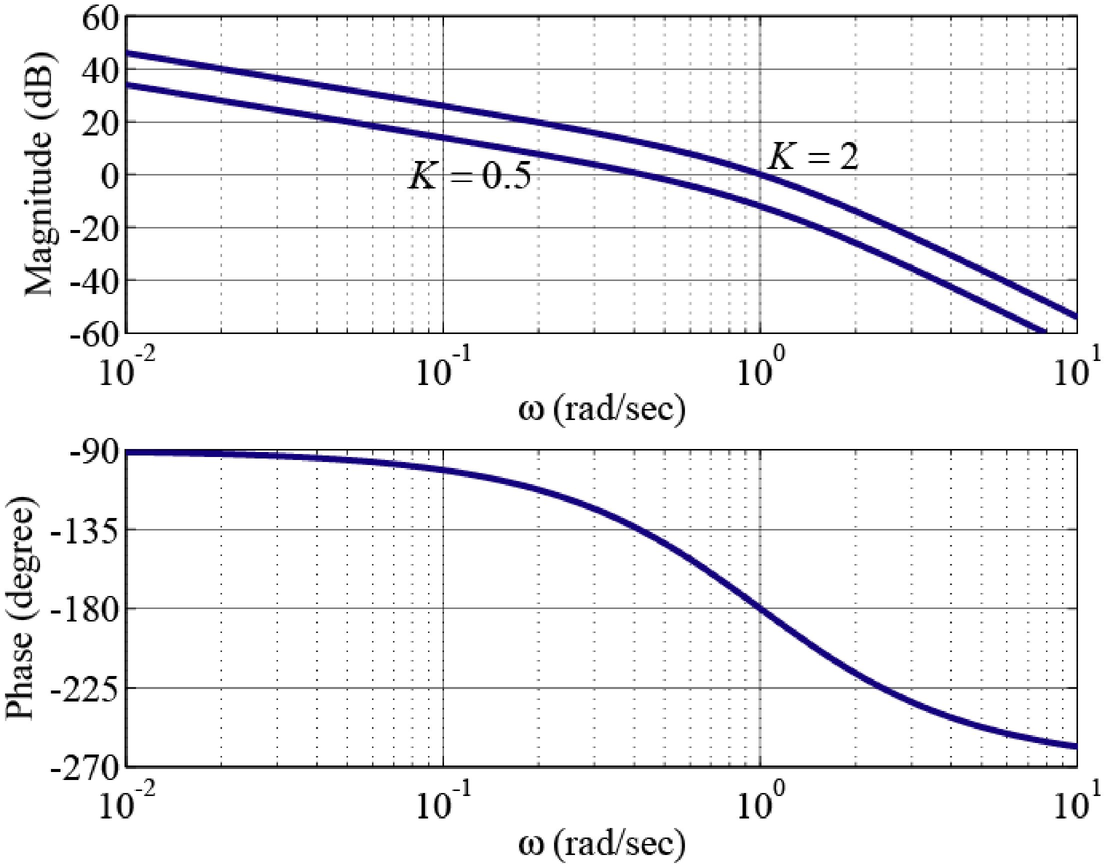




Stability Margin

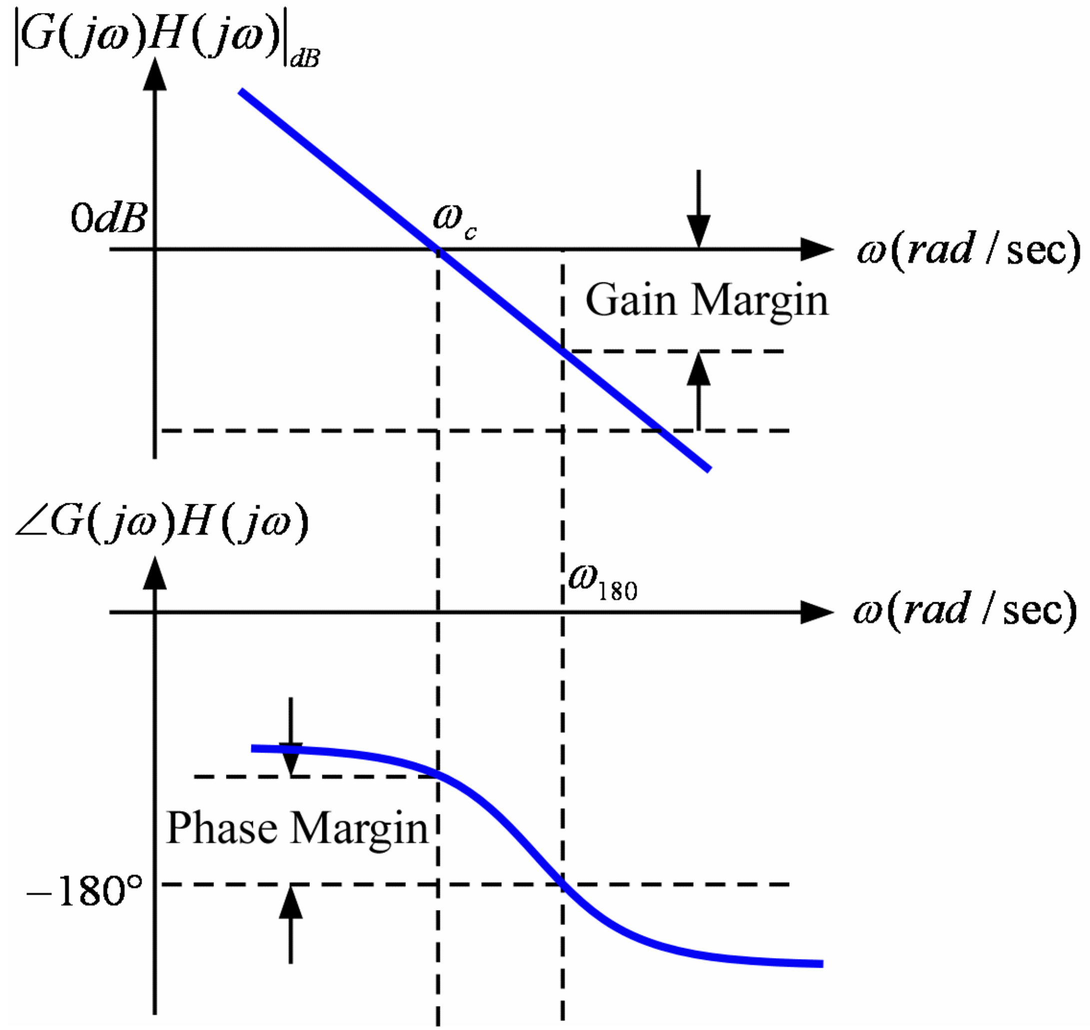


Stability Margin



Stability Margin

- Gain Margin
- Phase Margin



Stability Margin

- Gain Margin

$$GM = 0dB - \left|G(j\omega_{180})H(j\omega_{180})\right|_{dB}$$

$$= 20\log_{10} 1 - 20\log_{10}\left|G(j\omega_{180})H(j\omega_{180})\right|$$

$$= 20\log_{10}\frac{1}{\left|G(j\omega_{180})H(j\omega_{180})\right|}$$

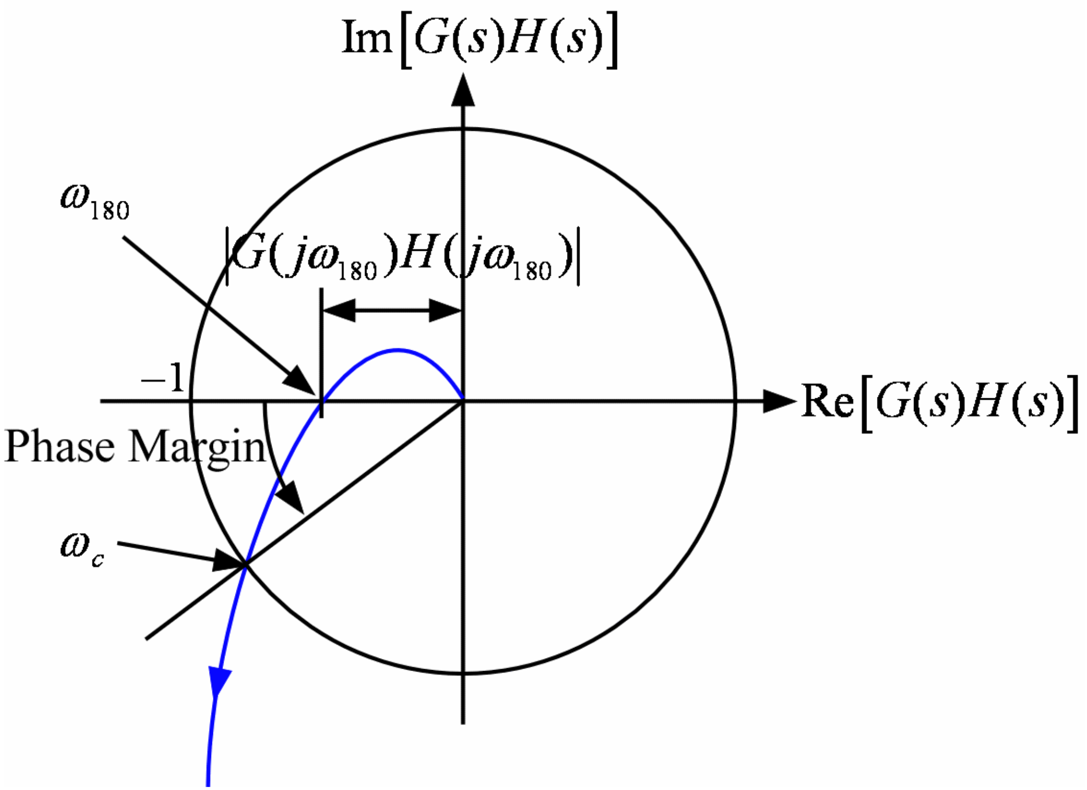
- Phase Margin

$$PM = 180^\circ + \angle\left(G(j\omega_c)H(j\omega_c)\right)$$

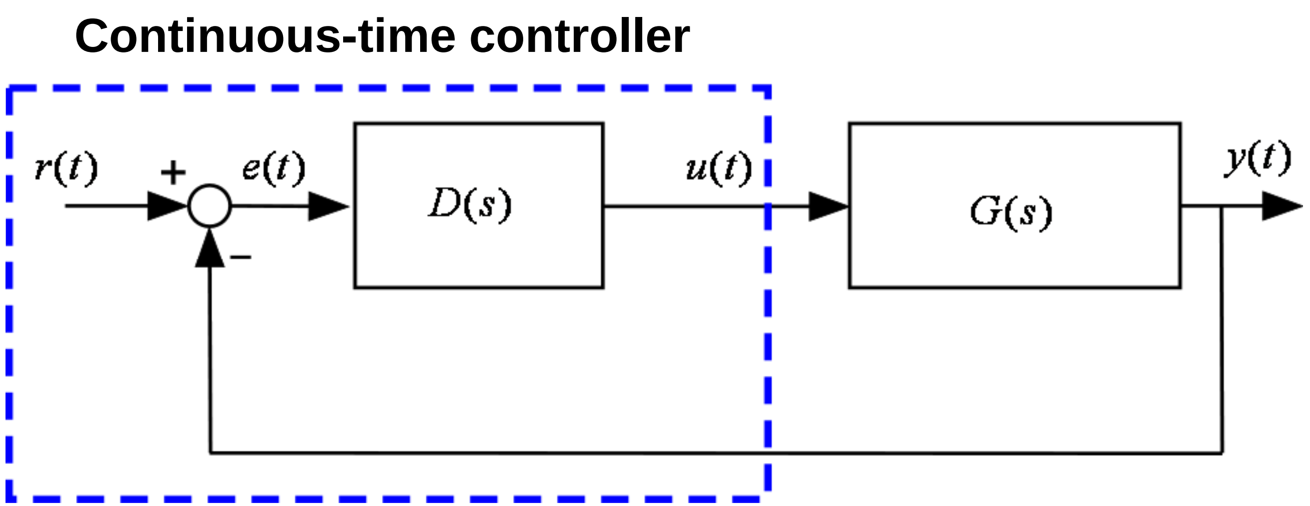
Stability Margin

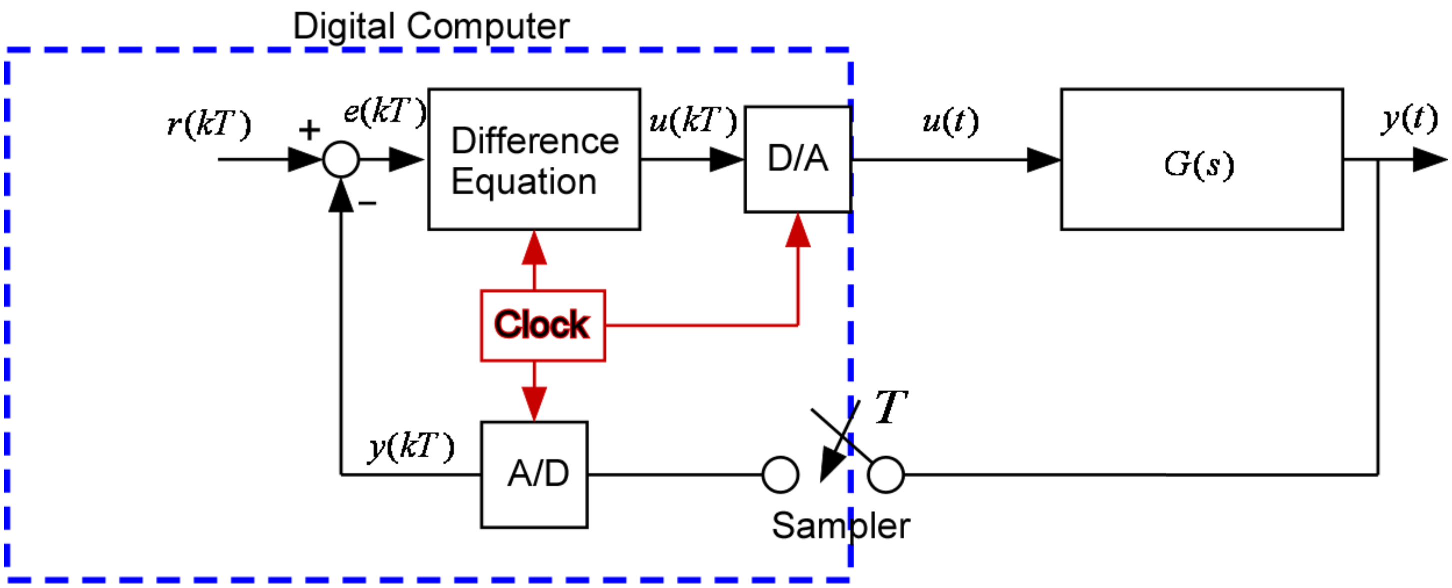
- Gain Margin

- Phase Margin



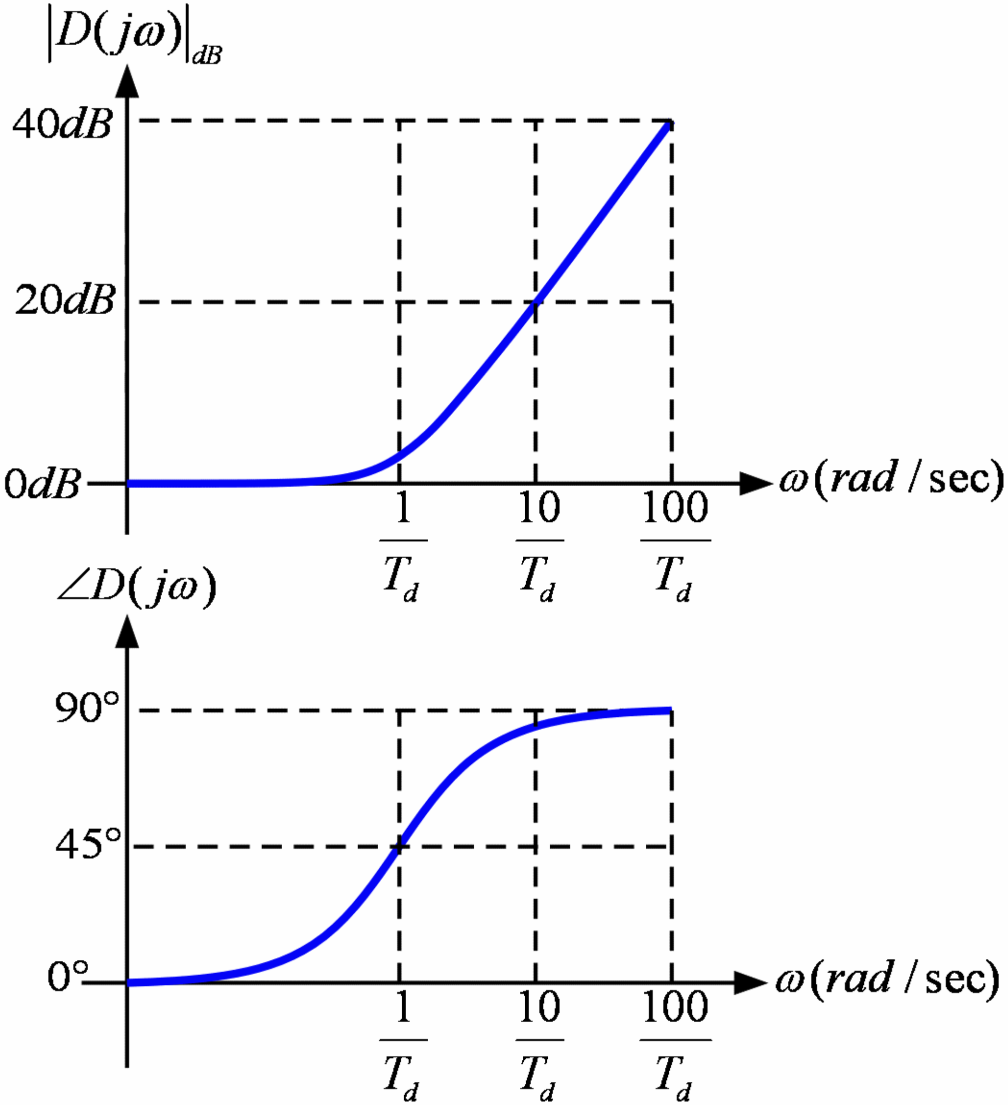
Control System





PD Controller

$$D(s) = K\left(1 + T_d s\right)$$



Example

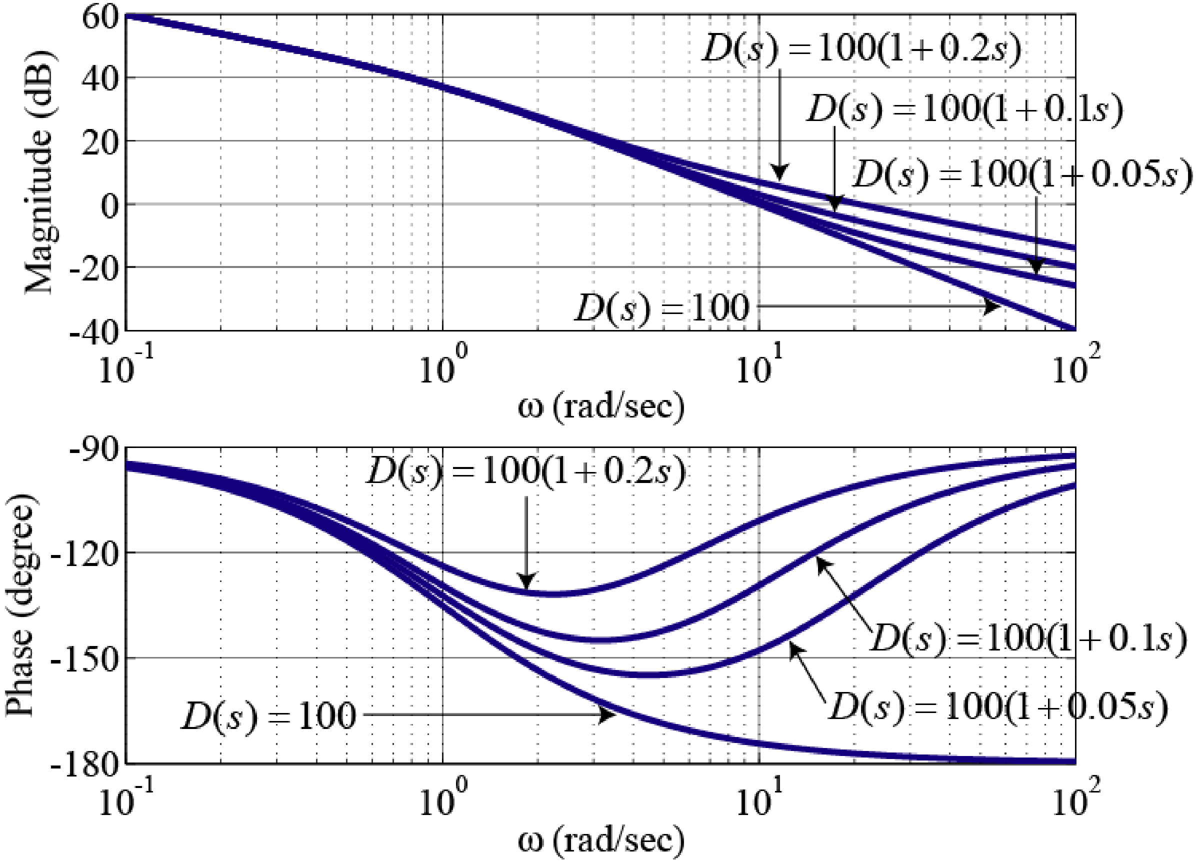
$$G(s) = \frac{1}{s(s+1)}$$

$$K_v = \lim_{s\to 0} s \frac{K}{s(s+1)} = K$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{K}$$

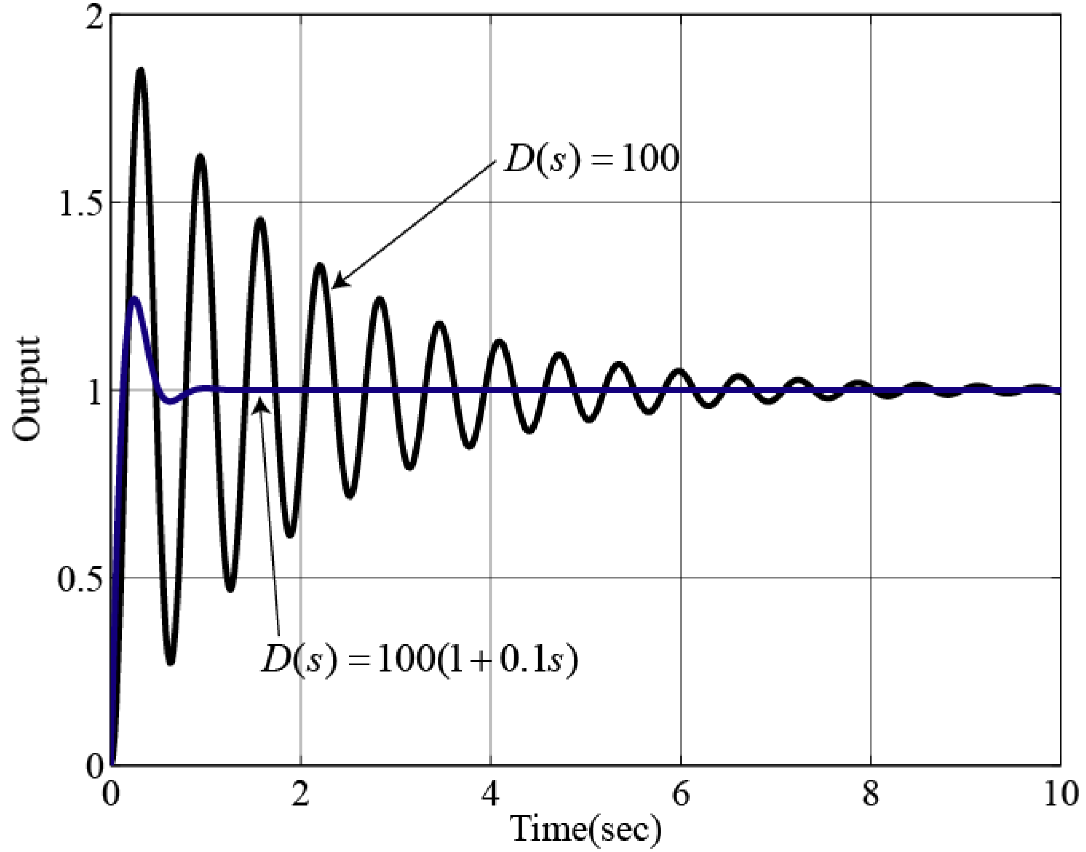
$$D(s) = 100(1 + 0.1s)$$

Example



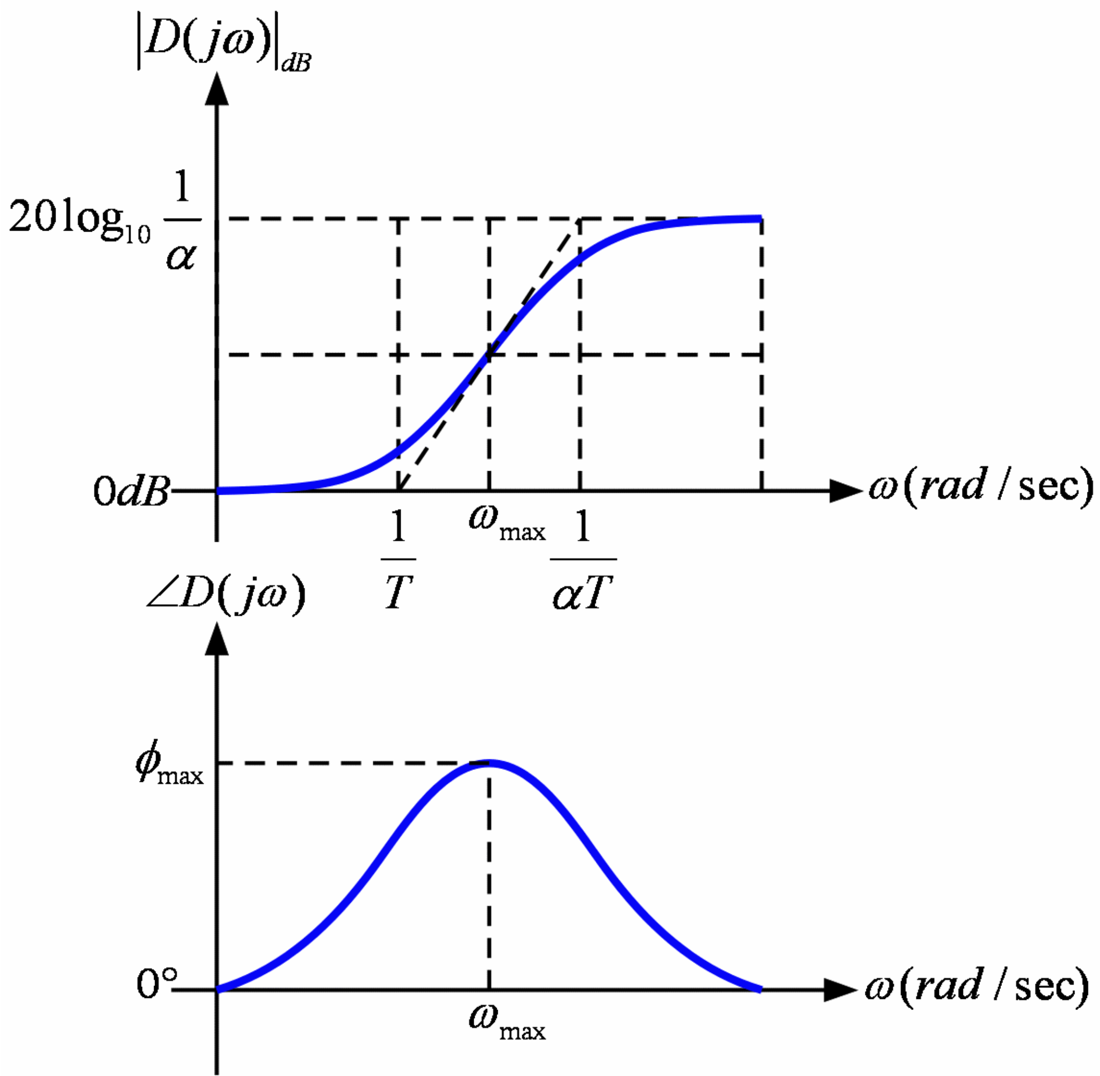
60
40
20
0
-20
-40
Magnitude (dB)
$D(s) = 100(1+0.2s)$
$D(s) = 100(1+0.1s)$
$D(s) = 100(1+0.05s)$
$D(s) = 100$
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)
-90
-120
-150
-180
Phase (degree)
$D(s) = 100(1+0.2s)$
$D(s) = 100(1+0.1s)$
$D(s) = 100(1+0.05s)$
$D(s) = 100$
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)

Example



Lead Compensator

$$D(s) = K\frac{Ts+1}{\alpha Ts+1}$$



Lead Compensator

$$\phi = \angle\left(\frac{jT\omega+1}{j\alpha T\omega+1}\right) = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega)$$

$$\log_{10}\omega_{\max} = \frac{1}{2}\left(\log_{10}\frac{1}{T} + \log_{10}\frac{1}{\alpha T}\right) \qquad \omega_{\max} = \frac{1}{T\sqrt{\alpha}}$$

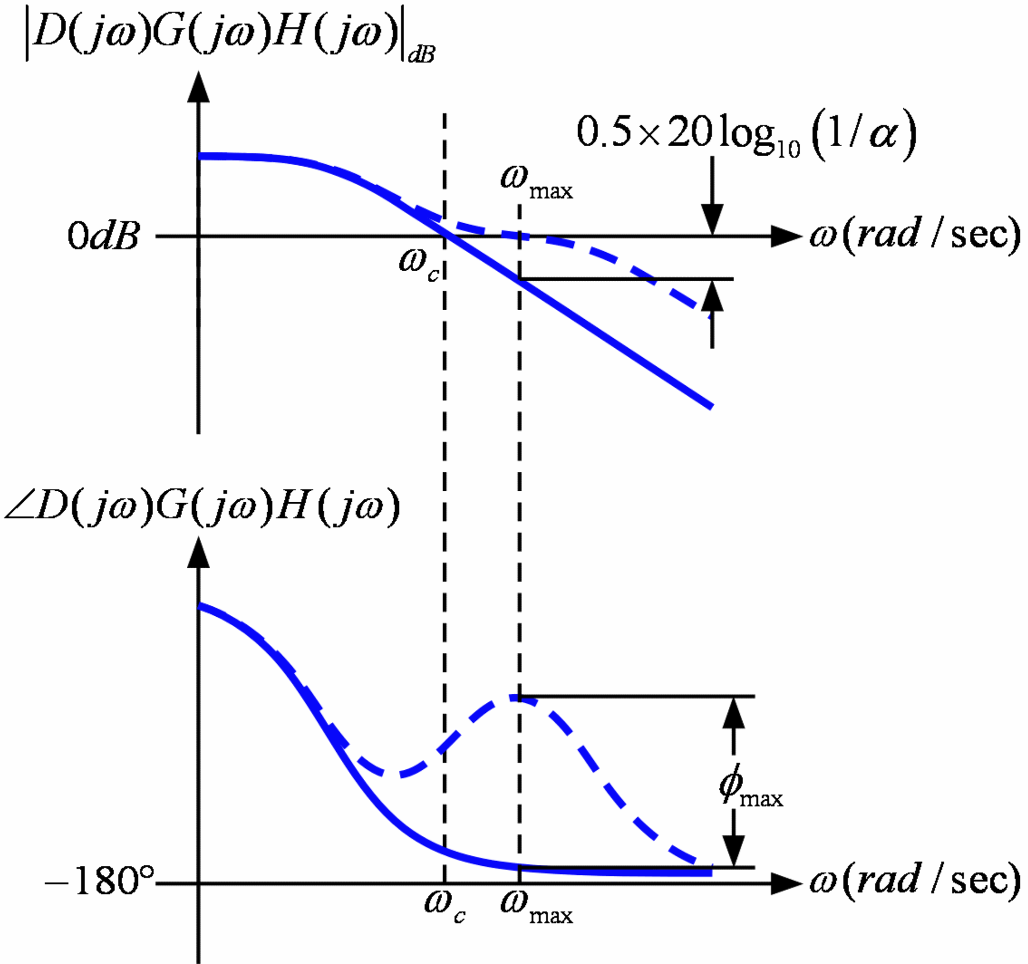
$$\phi_{\max} = \tan^{-1}\frac{1}{\sqrt{\alpha}} - \tan^{-1}\sqrt{\alpha}$$

$$\tan\phi_{\max} = \frac{1-\alpha}{2\sqrt{\alpha}} \qquad \sin\phi_{\max} = \frac{1-\alpha}{1+\alpha} \qquad \alpha = \frac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}}$$

Lead Compensator

$$20\log_{10}\left|KG(j\omega_{\max})H(j\omega_{\max})\right| = -0.5\times 20\log_{10}\frac{1}{\alpha}$$

$$T = \frac{1}{\omega_{\max}\sqrt{\alpha}}$$



Example

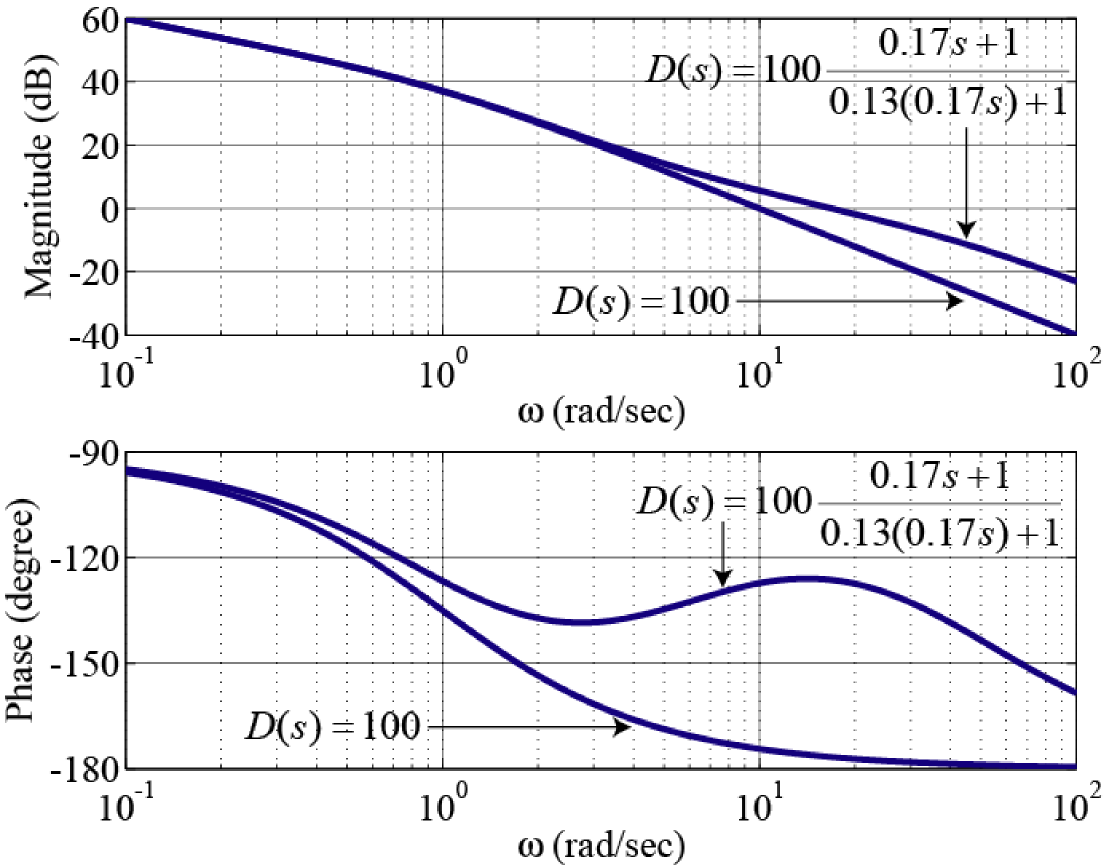
$$\phi_{\max} = 50^\circ \qquad \alpha = 0.13$$

$$0.5 \times 20\log_{10}\left(1/\alpha\right) = 9dB$$

$$T = \frac{1}{\omega_{\max}\sqrt{\alpha}} = \frac{1}{16.7\sqrt{0.13}} = 0.17$$

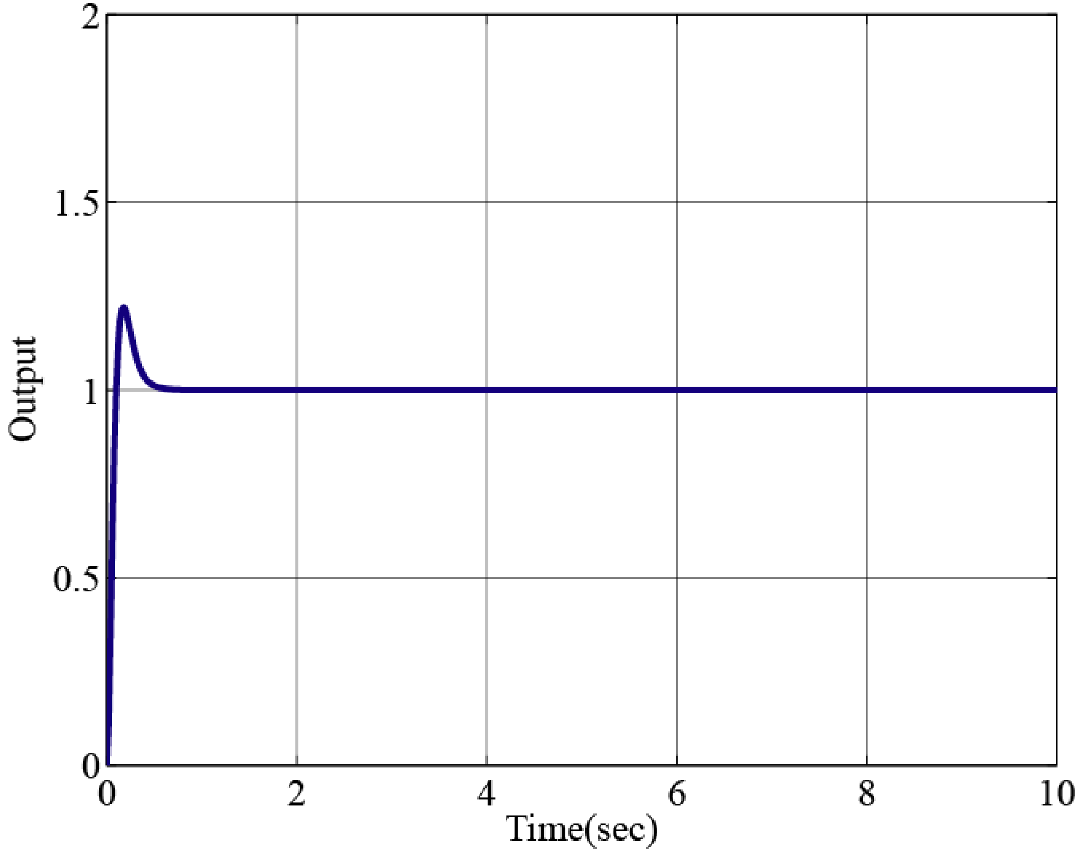
$$D(s) = K\frac{Ts+1}{\alpha Ts+1} = 100\frac{0.17s+1}{0.13(0.17s)+1}$$

Example



Magnitude (dB)
60
40
20
0
-20
-40
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)
$D(s) = 100\frac{0.17s+1}{0.13(0.17s)+1}$
$D(s) = 100$
Phase (degree)
-90
-120
-150
-180
10^{-1}
10^{0}
10^{1}
10^{2}
ω (rad/sec)
$D(s) = 100\frac{0.17s+1}{0.13(0.17s)+1}$
$D(s) = 100$

Example



MATLAB lead.m

```
num=100;
den=[1 1 0];
G=tf(num,den)

u=linspace(1,1,200);
t=linspace(0,10,200);

[y]=lsim(feedback(G,1),u,t);
figure(1)
plot(t,y);
grid on

w=logspace(0,3,200);
[mag,phase]=bode(num,den,w);
[gm,pm,wcg,wcp]=margin(G)
figure(2)
margin(num,den)
```

MATLAB lead.m

```
phimax=50;
alpha=(1-sin(pi*phimax/180))/(1+sin(pi*phimax/180))
10*log10(1/alpha)

[w' 20*log10(mag) phase ]

wmax=16.5;
T=1/(wmax*sqrt(alpha));
num1=[T 1];
den1=[T*alpha 1];
```

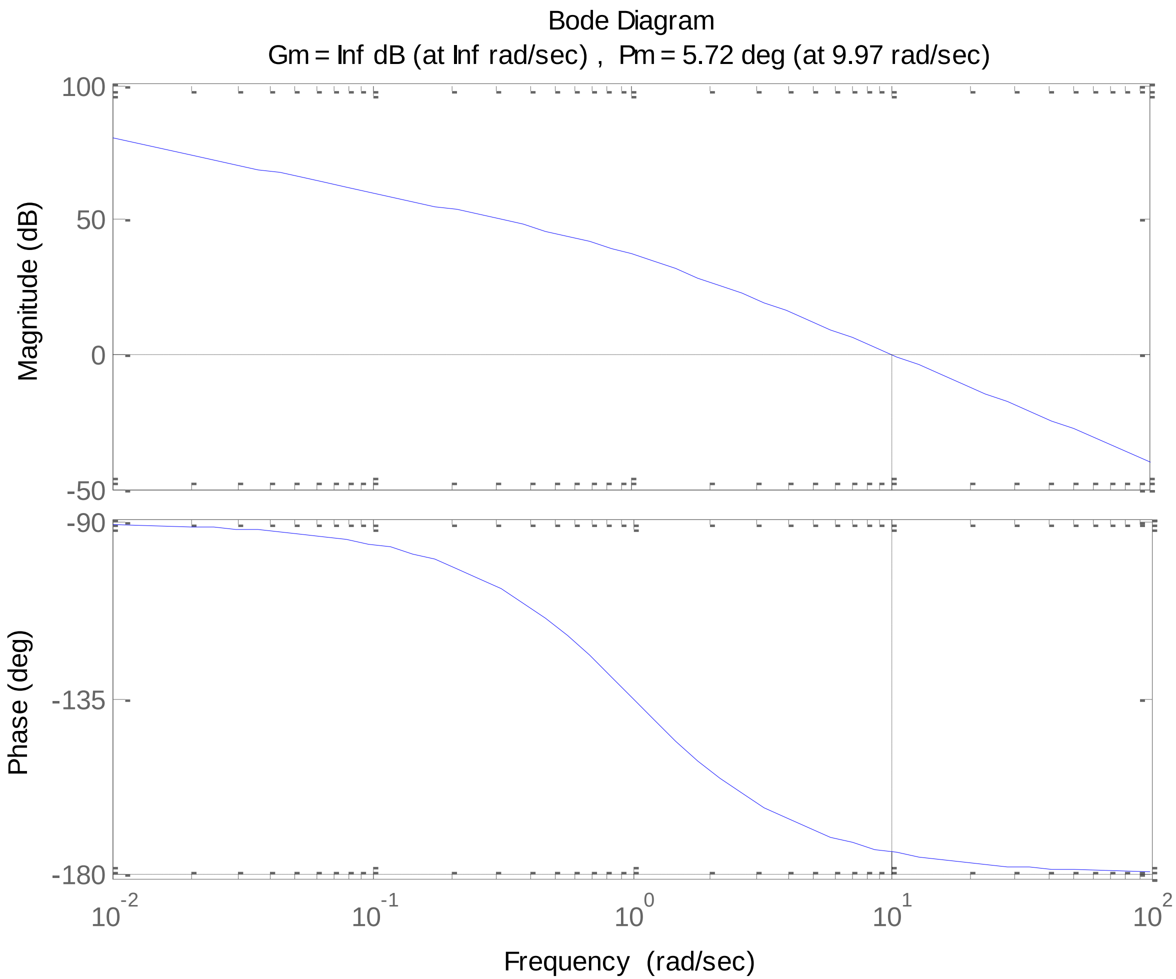
MATLAB lead.m

```
num=conv(num1,num);
den=conv(den1,den);
[mag,phase]=bode(num,den,w);
[gm,pm,wcg,wcp]=margin(mag,phase,w)
figure(3)
margin(num,den)

[y]=lsim(feedback(tf(num,den),1),u,t);
figure(4)
plot(t,y);
grid on

D=tf(num1,den1)
Dz=c2d(D,1/2000,'tustin')
```

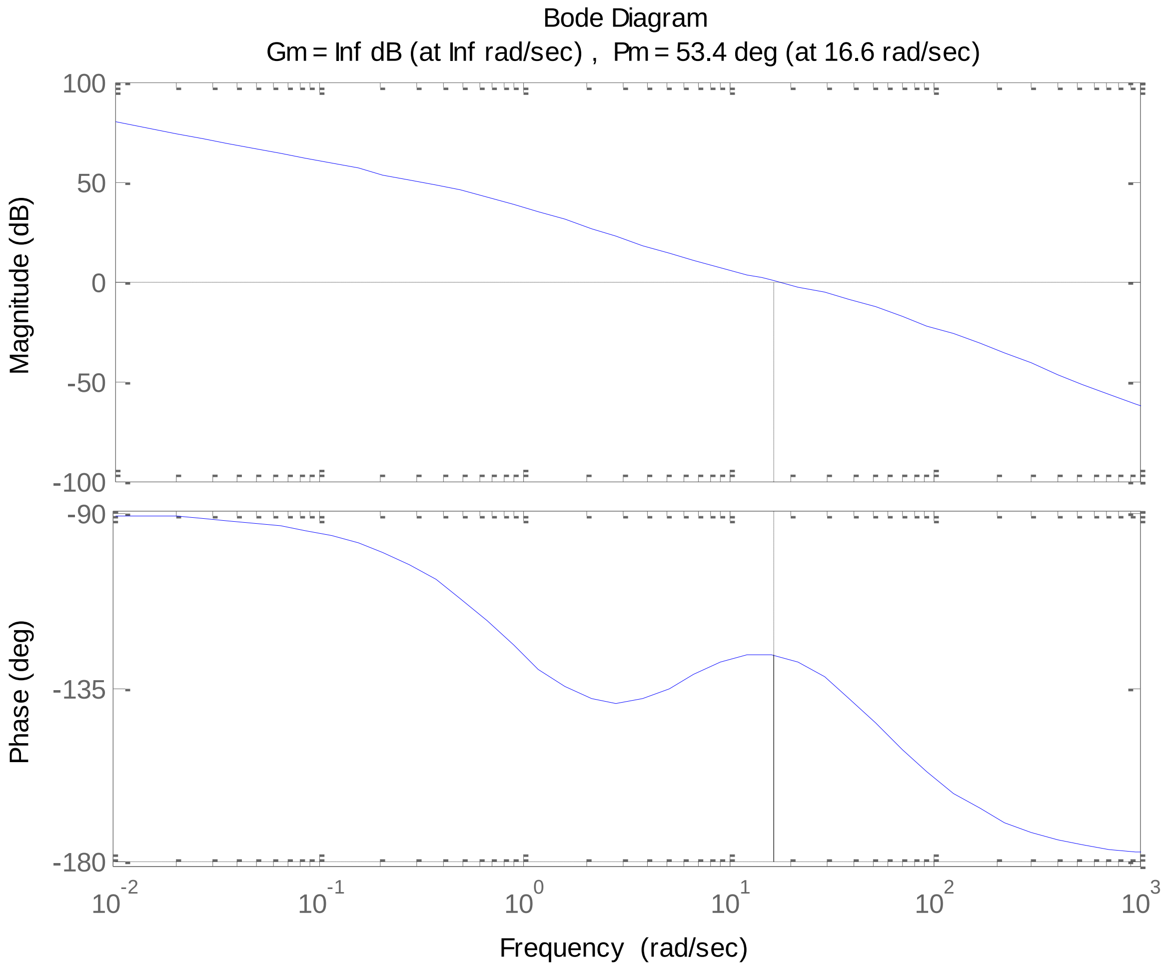
MATLAB lead.m

```
alpha =

  1.3247e-001


ans =

  8.7787e+000


ans =

  1.0000e+000  3.6990e+001 -1.3500e+002


  1.4481e+001 -6.4528e+000 -1.7605e+002
  1.4993e+001 -7.0545e+000 -1.7618e+002
  1.5522e+001 -7.6562e+000 -1.7631e+002
  1.6071e+001 -8.2580e+000 -1.7644e+002
  1.6638e+001 -8.8599e+000 -1.7656e+002
  1.7226e+001 -9.4618e+000 -1.7668e+002
```

MATLAB lead.m



MATLAB lead.m



Digital Implementation

- Tustin's Method

$$D(z) = \frac{U(z)}{E(z)} = K\frac{Ts+1}{\alpha Ts+1}\bigg|_{s=\frac{2}{T_s}\frac{z-1}{z+1}} = K\frac{T\frac{2}{T_s}\frac{z-1}{z+1}+1}{\alpha T\frac{2}{T_s}\frac{z-1}{z+1}+1}$$

$$u(k) = \frac{1}{T_s+2\alpha T}\left[K\left(T_s+2T\right)e(k)+K\left(T_s-2T\right)e(k-1)+\left(2\alpha T-T_s\right)u(k-1)\right]$$