
State-Space Design

Continuous-Time System

Full-State Feedback

- Zero reference input: regulator

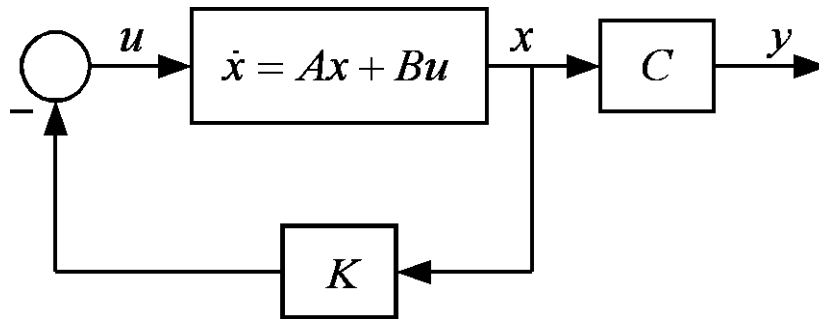
$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$u = -Kx = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$= -(k_1 x_1 + k_2 x_2 + \cdots + k_n x_n)$$

Full-State Feedback

$$\dot{x} = Ax - BKx = (A - BK)x$$



$$|sI - (A - BK)| = 0$$

$$|sI - (A - BK)| = (s - r_1)(s - r_2) \cdots (s - r_n)$$

Example

$$G(s) = \frac{1}{s(s+1)} \quad D(s) = 100(1+0.1s)$$

$$1 + D(s)G(s) = 1 + \frac{100(1+0.1s)}{s(s+1)} = 0 \quad s^2 + 11s + 100 = 0$$

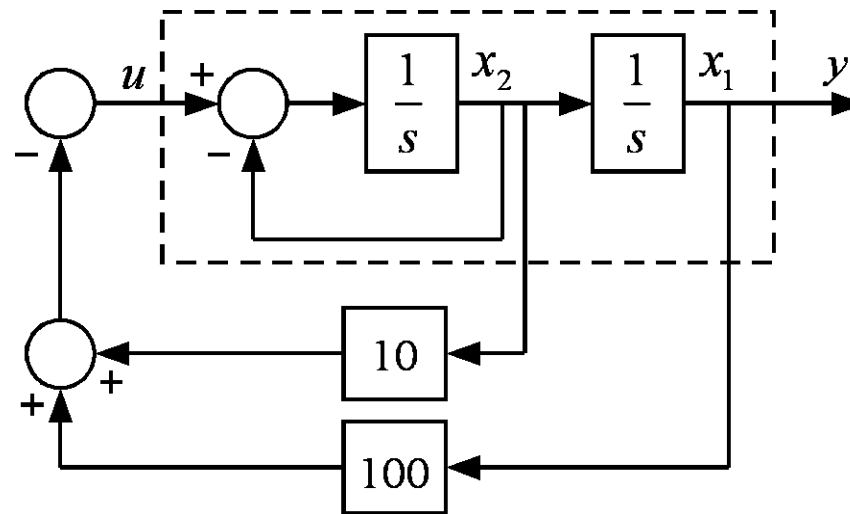
$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{aligned} |sI - (A - BK)| &= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] \right) \right| \\ &= \left| \begin{bmatrix} s & -1 \\ k_1 & s+1+k_2 \end{bmatrix} \right| = s^2 + (1+k_2)s + k_1 \end{aligned}$$

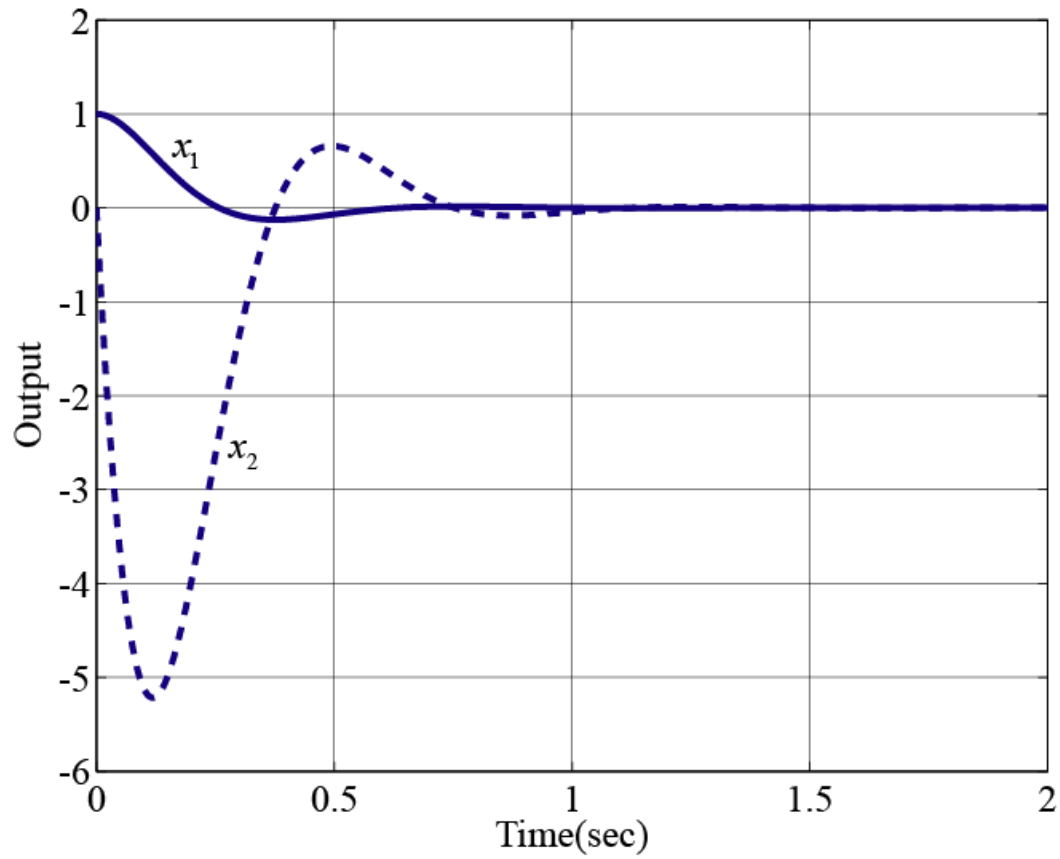
Example

$$k_1 = 100, k_2 = 10$$

$$u = -(100x_1 + 10x_2)$$



Example



Control Canonical Form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 = 0$$

$$\alpha_c(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0 = 0$$

Control Canonical Form

$$\dot{x} = (A - BK)x$$

$$= \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \right) x$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 - k_1 & -a_1 - k_2 & -a_2 - k_3 & \cdots & -a_{n-1} - k_n \end{bmatrix} x$$

Control Canonical Form

$$s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_1 + k_2)s + (a_0 + k_1) = 0$$

$$k_1 = \alpha_0 - a_0$$

$$k_2 = \alpha_1 - a_1$$

$$\vdots$$

$$k_n = \alpha_{n-1} - a_{n-1}$$

Ackermann's formula

$$K = [0 \quad \cdots \quad 0 \quad 1] M_C^{-1} \alpha_c(A)$$

$$M_C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$\alpha_c(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1A + \alpha_0I$$

Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] x$$

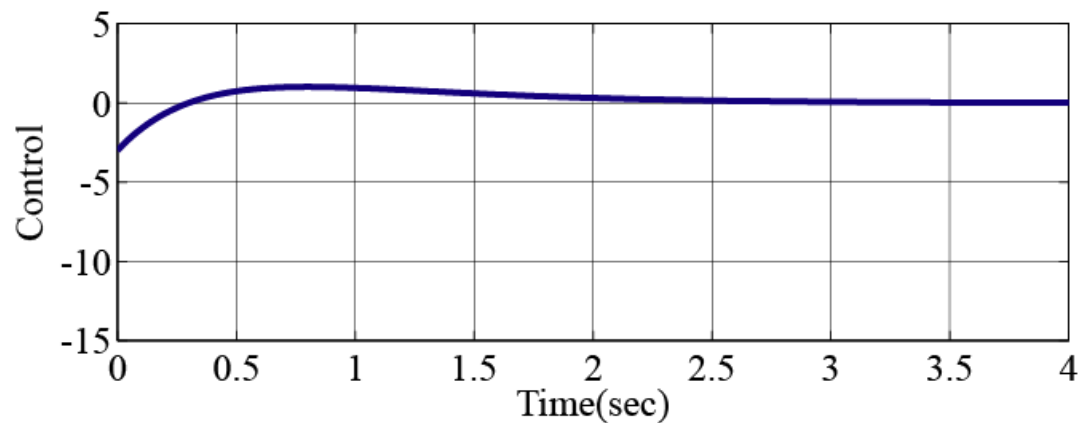
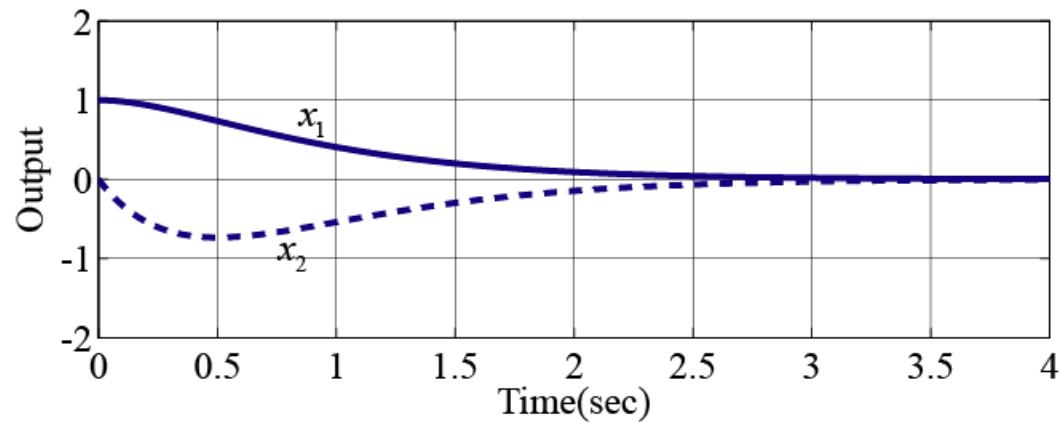
$$\alpha_c(s) = (s + 2)^2 = s^2 + 4s + 4 = 0$$

$$M_c = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\alpha_c \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 + 4 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

Example

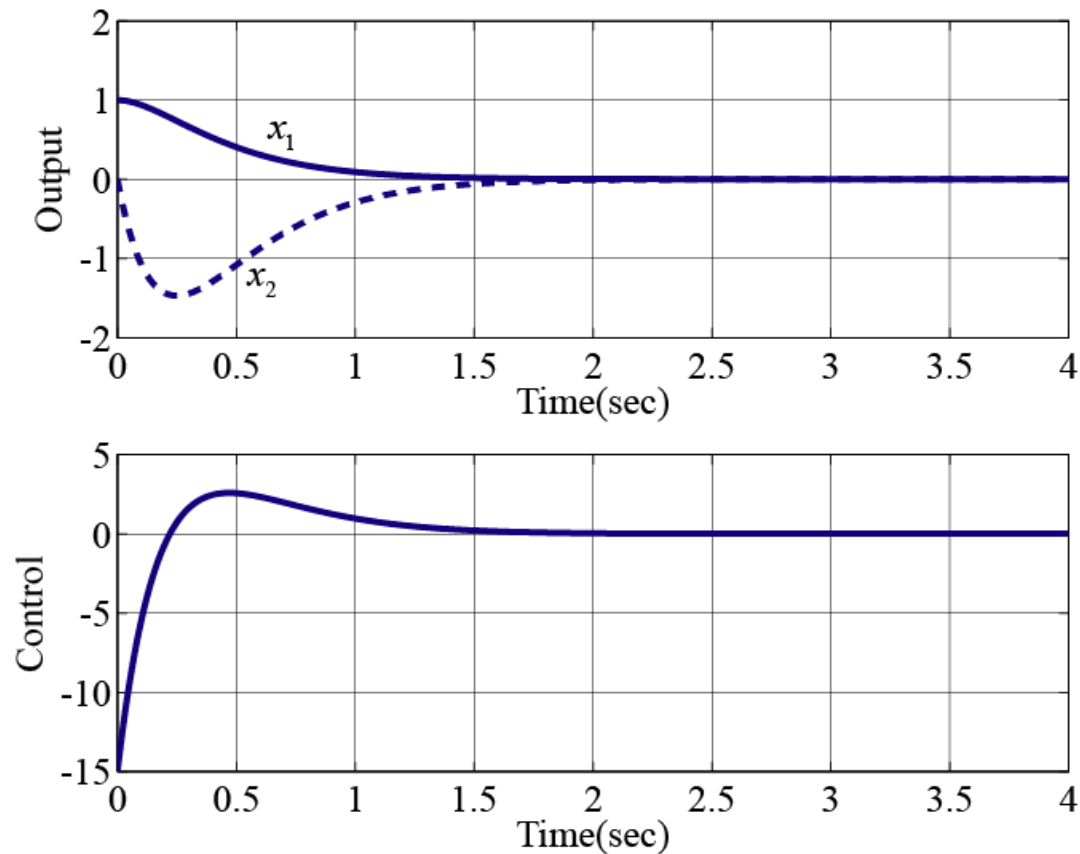
$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \end{bmatrix} \quad u = -(3x_1 + 4x_2)$$



Example

$$\alpha_c(s) = (s + 4)^2 = s^2 + 8s + 16 = 0$$

$$u = -(15x_1 + 8x_2)$$

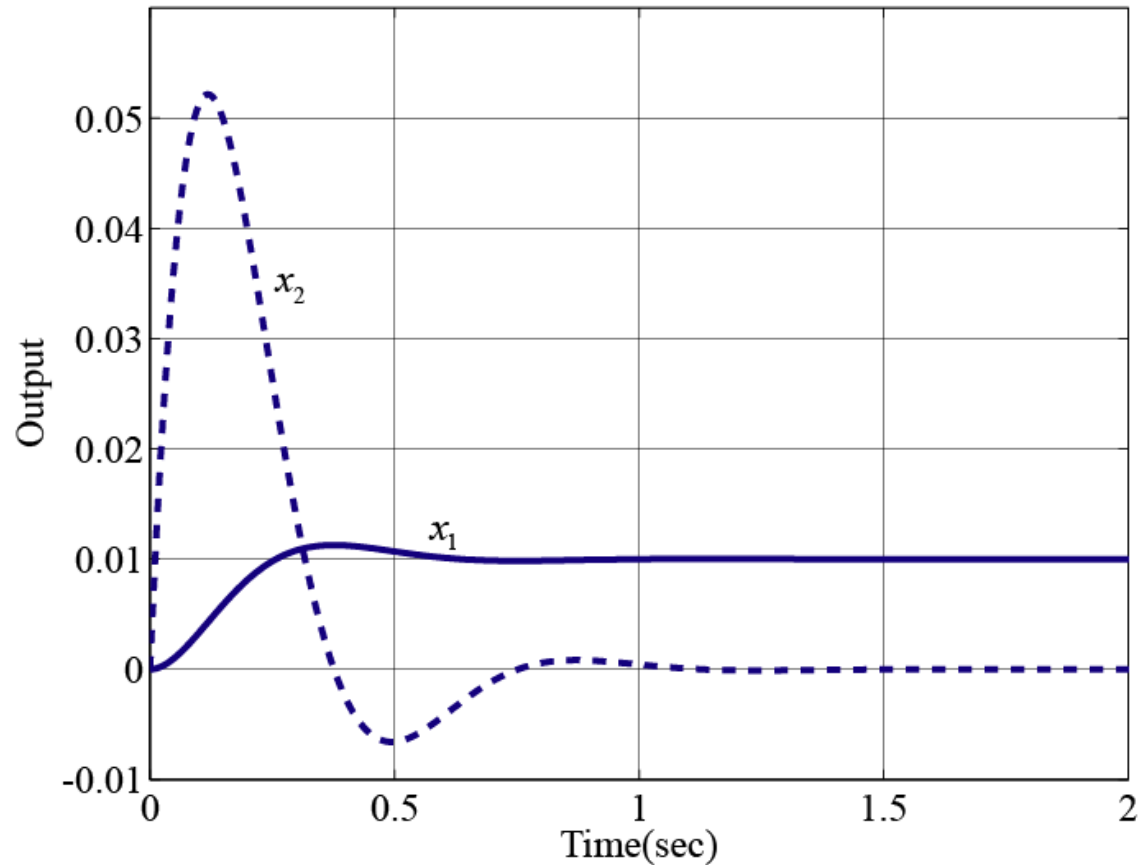


Full-State Feedback

- Non-zero reference input

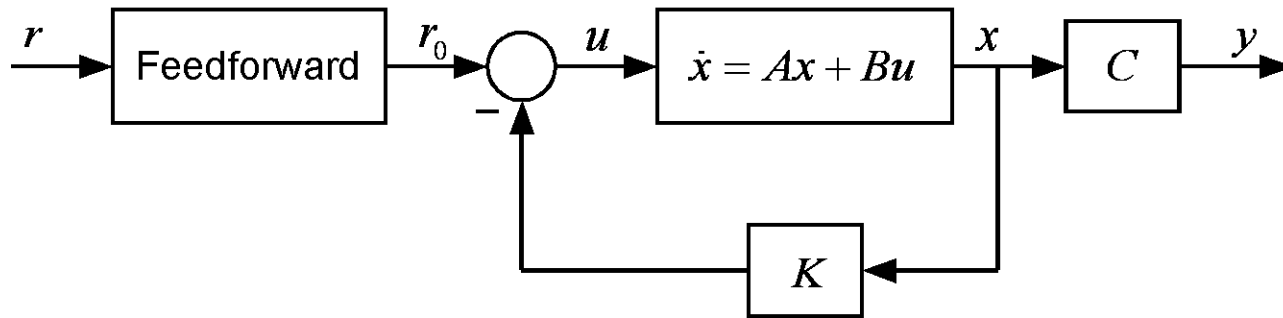
$$u = r_0 - Kx$$

$$u = r_0 - (100x_1 + 10x_2)$$



Full-State Feedback

- Non-zero reference input



$$\frac{R_0(s)}{R(s)} = D_F(s)$$

$$\dot{x} = Ax + Bu = Ax + B(r_0 - Kx) = (A - BK)x + Br_0$$

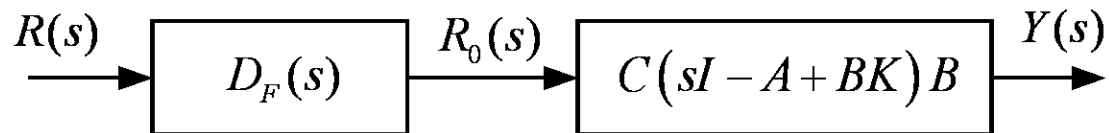
$$X(s) = (sI - A + BK)^{-1} BR_0(s)$$

Full-State Feedback

- Non-zero reference input

$$\frac{Y(s)}{R_0(s)} = C(sI - A + BK)^{-1} B$$

$$\frac{Y(s)}{R(s)} = C(sI - A + BK)^{-1} B D_F(s)$$



Example: Non-zero reference input

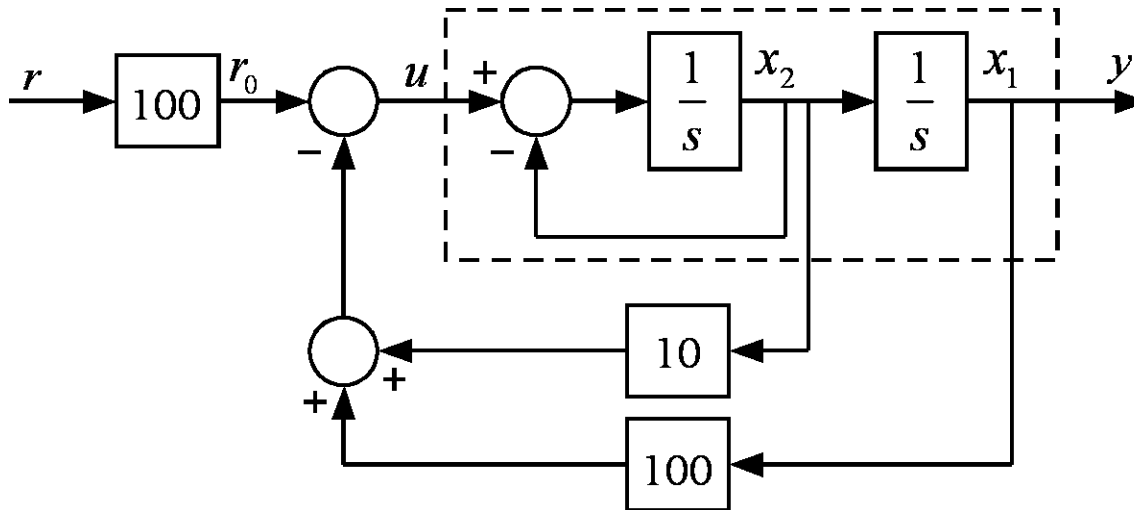
$$\begin{aligned}\frac{Y(s)}{R_0(s)} &= C (sI - A + BK)^{-1} B \\ &= [1 \quad 0] \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [100 \quad 10] \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 + 11s + 100}\end{aligned}$$

$$D_F(s) = 100$$

$$\frac{Y(s)}{R(s)} = \frac{Y(s)}{R_0(s)} \cdot \frac{R_0(s)}{R(s)} = \frac{Y(s)}{R_0(s)} \cdot D_F(s) = \frac{100}{s^2 + 11s + 100}$$

Example: Non-zero reference input

$$u(t) = 100r - (100x_1 + 10x_2)$$



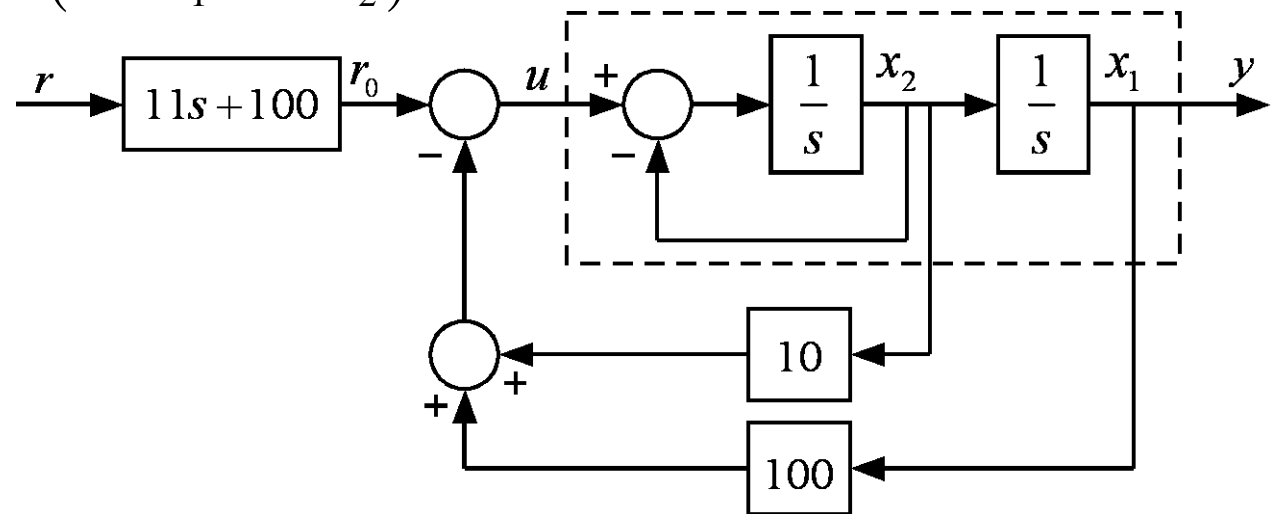
Example: Non-zero reference input

- System Type 2

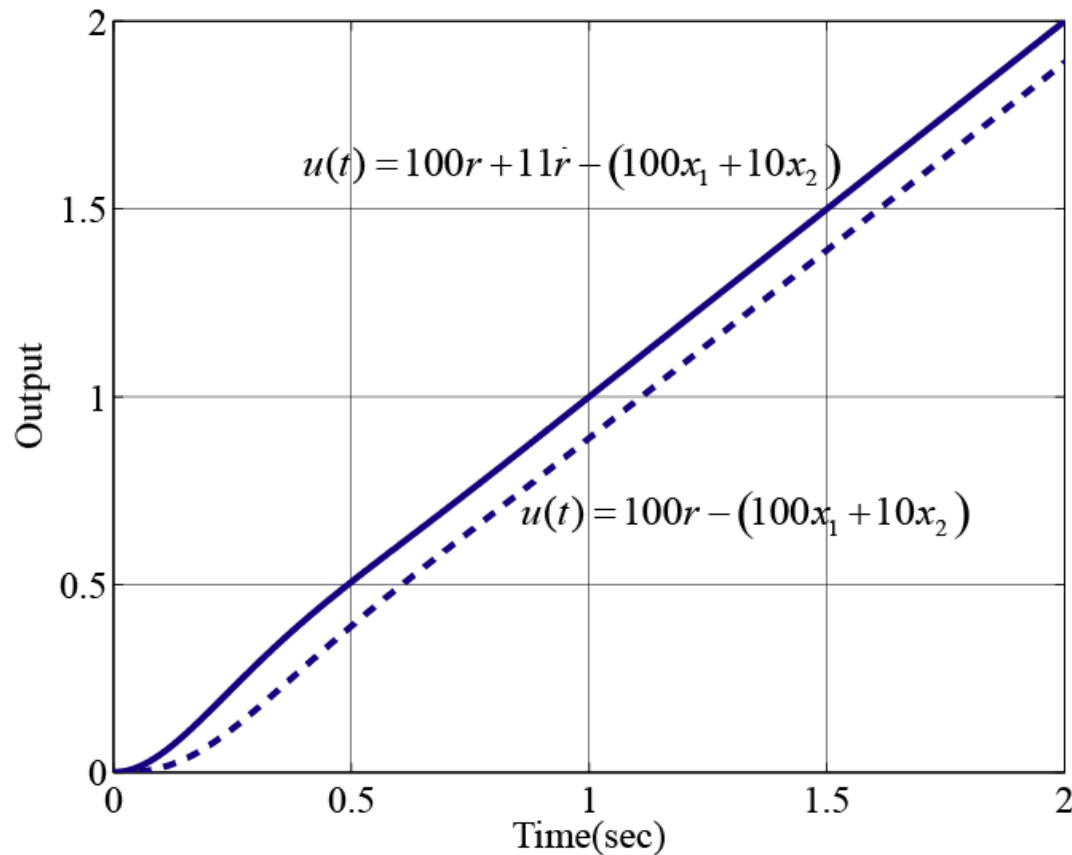
$$\frac{Y(s)}{R(s)} = \frac{Y(s)}{R_0(s)} \cdot D_F(s) = \frac{D_F(s) \cdot 1}{s^2 + 11s + 100} = \frac{11s + 100}{s^2 + 11s + 100}$$

$$D_F(s) = 11s + 100$$

$$u(t) = 100r + 11\dot{r} - (100x_1 + 10x_2)$$



Example: Non-zero reference input



State Estimator

$$\dot{\hat{x}} = A_e \hat{x} + B_e u + Ly$$

$$\hat{x}(t) \rightarrow x(t), t \rightarrow \infty$$

$$e = x - \hat{x} \quad e \rightarrow 0, t \rightarrow \infty$$

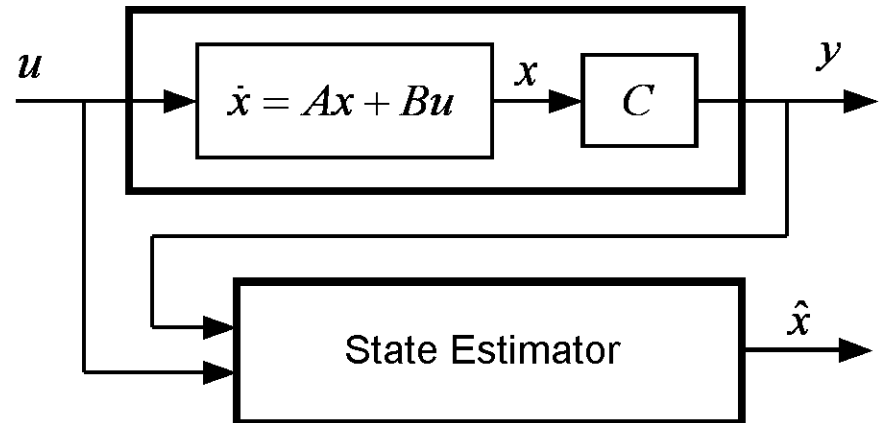
$$\dot{x} - \dot{\hat{x}} = Ax + Bu - A_e \hat{x} - B_e u - Ly$$

$$= Ax + Bu - A_e \hat{x} - B_e u - LCx$$

$$= (A - LC)x - A_e \hat{x} + (B - B_e)u$$

$$A_e = A - LC \quad B_e = B \quad \dot{e} = (A - LC)e$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly = A\hat{x} + Bu + L(y - C\hat{x})$$



State Estimator

$$\begin{aligned}\alpha_e(s) &= (s - \beta_1)(s - \beta_2) \cdots (s - \beta_n) \\ &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0 = 0\end{aligned}$$

$$|sI - (A - LC)| = 0$$

$$A \leftrightarrow A^T$$

$$(A - LC)^T = (A^T - C^T L^T)$$

$$B \leftrightarrow C^T$$

$$K \leftrightarrow L^T$$

$$L^T = [0 \quad \cdots \quad 0 \quad 1] (M_o^T)^{-1} \alpha_e(A^T)$$

$$M_o^T = \begin{bmatrix} C^T & C^T A^T & C^T (A^T)^2 & \cdots & C^T (A^T)^{n-1} \end{bmatrix}$$

$$\alpha_e(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1A + \alpha_0I$$

$$L = \alpha_e(A)M_o^{-1} [0 \quad \cdots \quad 0 \quad 1]^T$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\alpha_e(s) = (s + 2)^2 = s^2 + 4s + 4 = 0$$

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} [1 \ 0] \\ [1 \ 0] \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

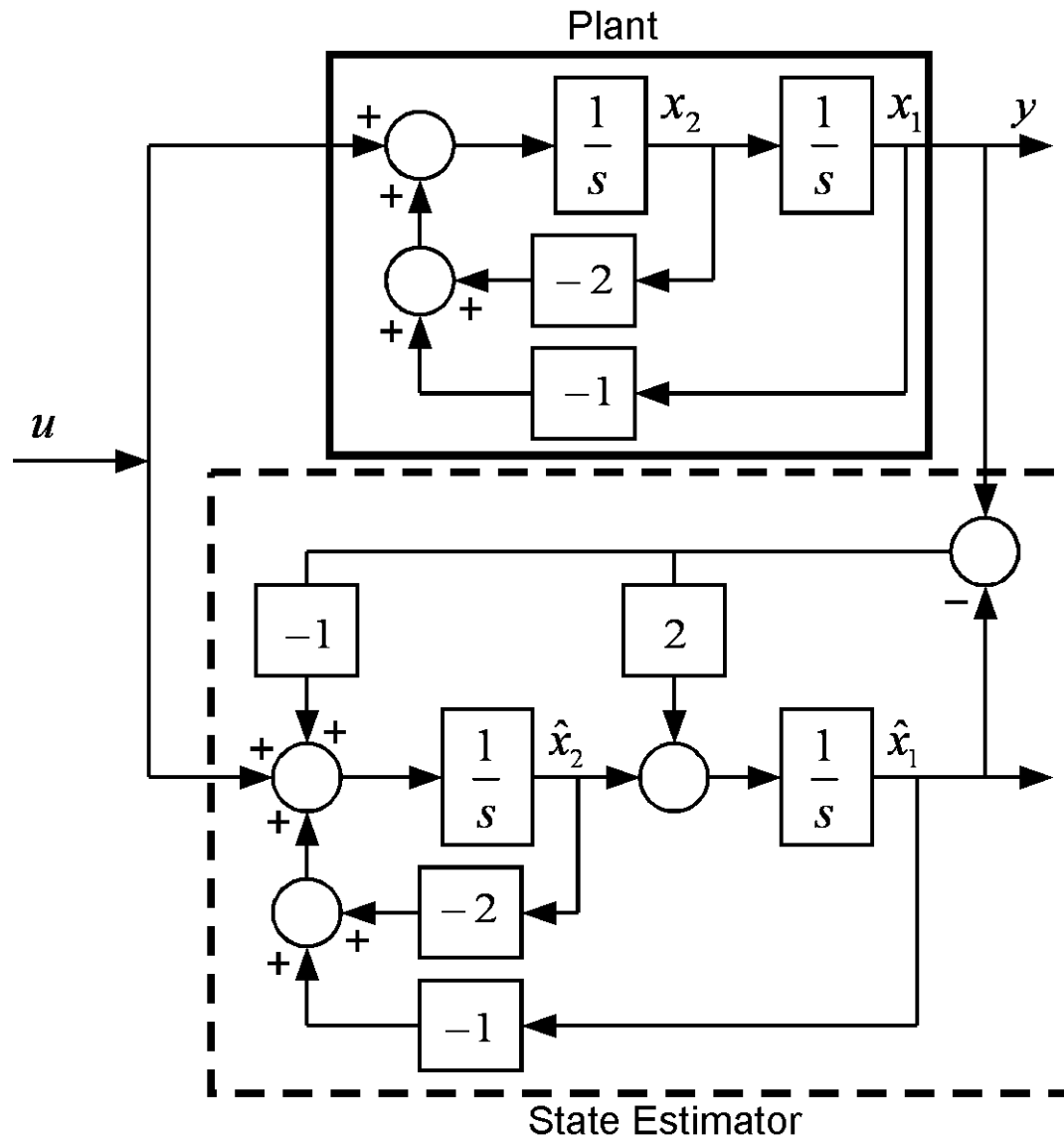
$$\alpha_e \left(\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}^2 + 4 \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$$

Example

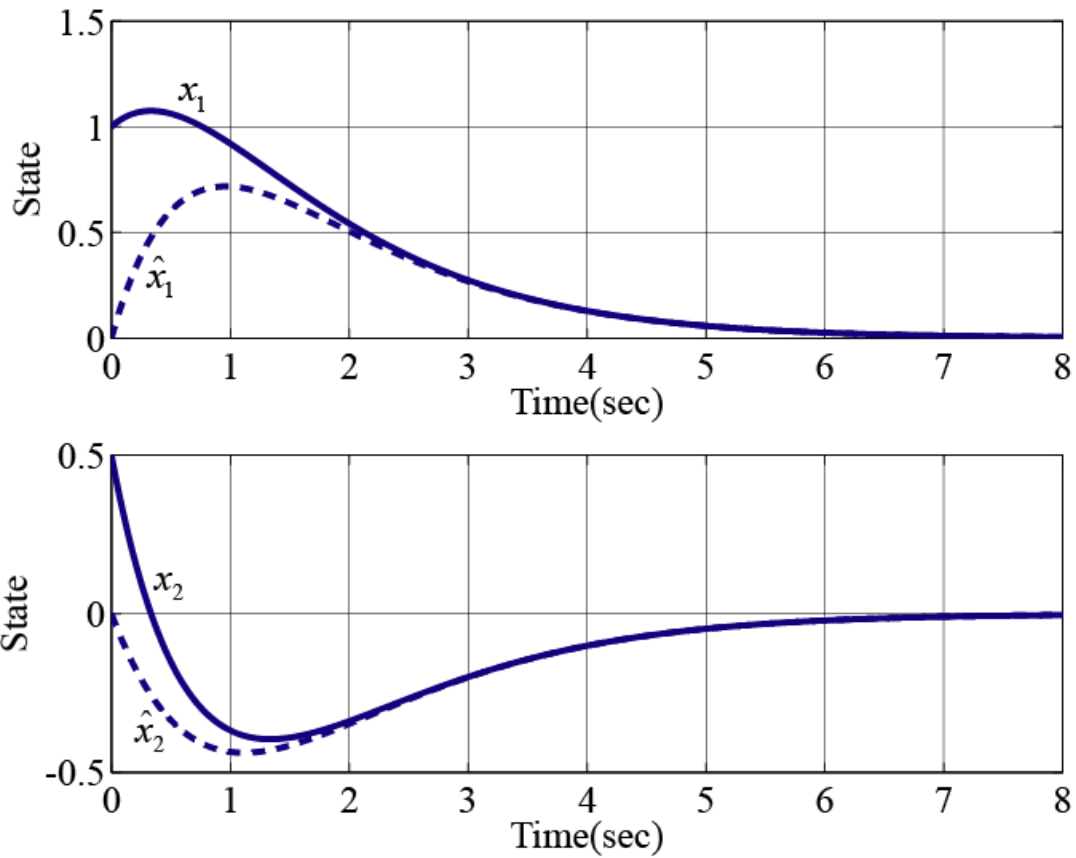
$$L = \begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 2 \\ -1 \end{bmatrix} (y - \hat{x}_1)$$

Example



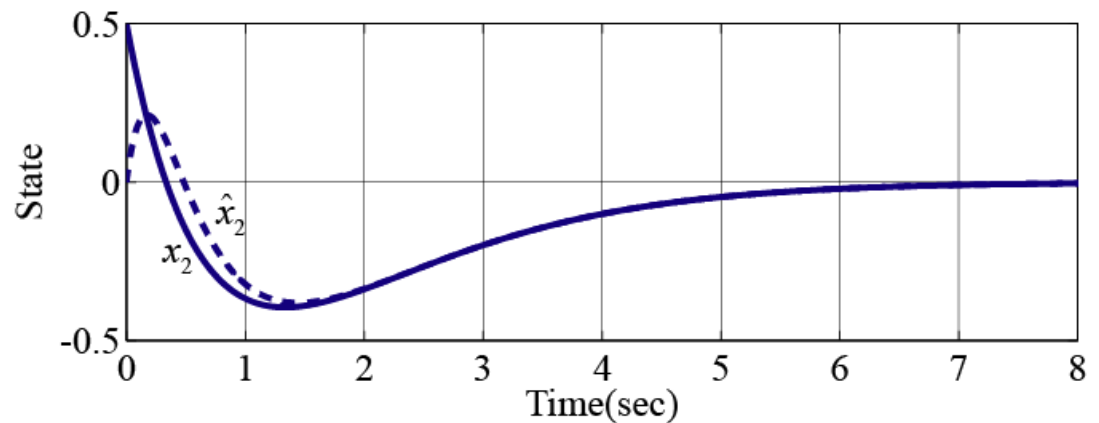
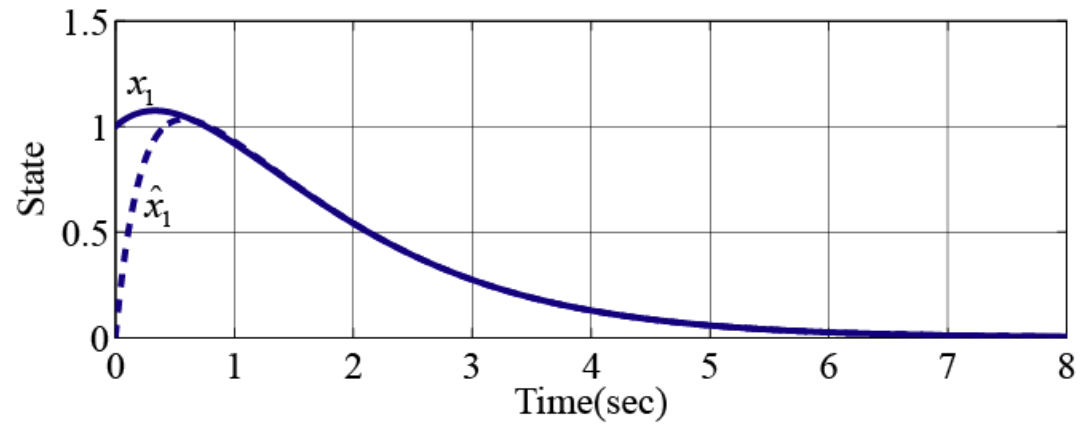
Example



Example

$$\alpha_e(s) = (s + 4)^2 = s^2 + 8s + 16 = 0$$

$$L = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$



Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

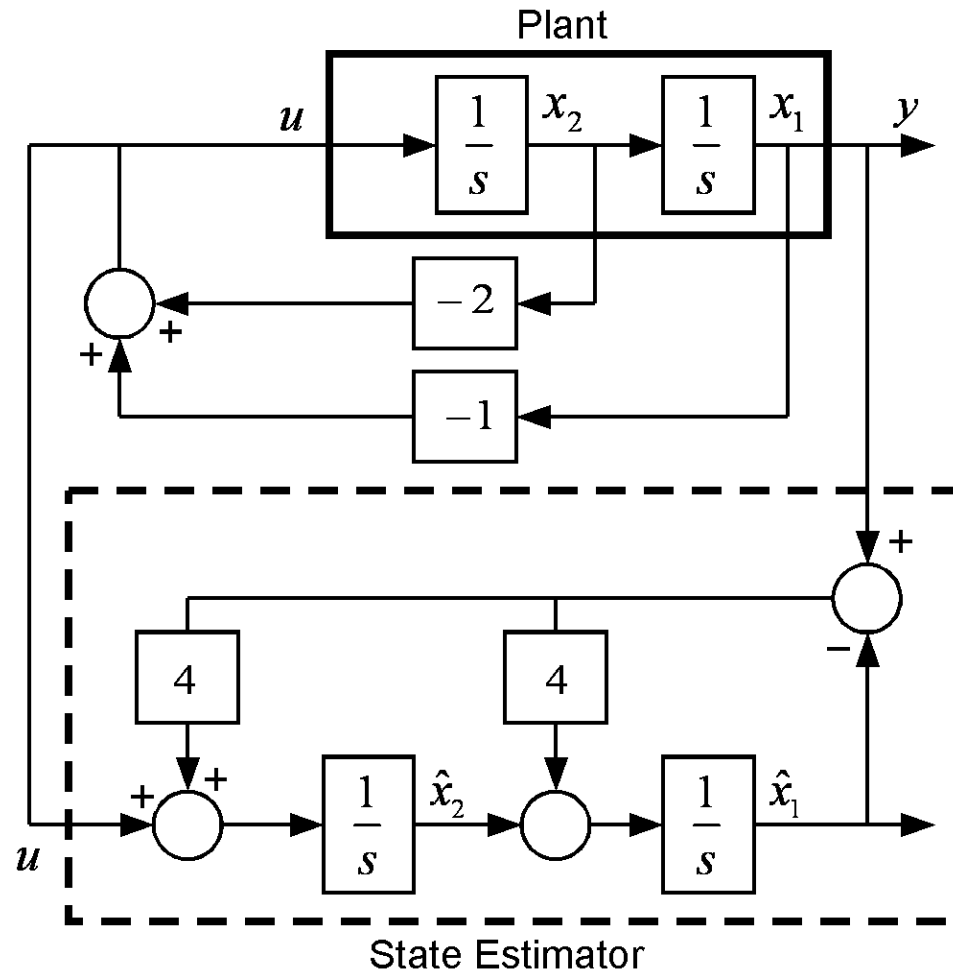
$$\alpha_e(s) = (s + 2)^2 = s^2 + 4s + 4 = 0$$

$$L = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

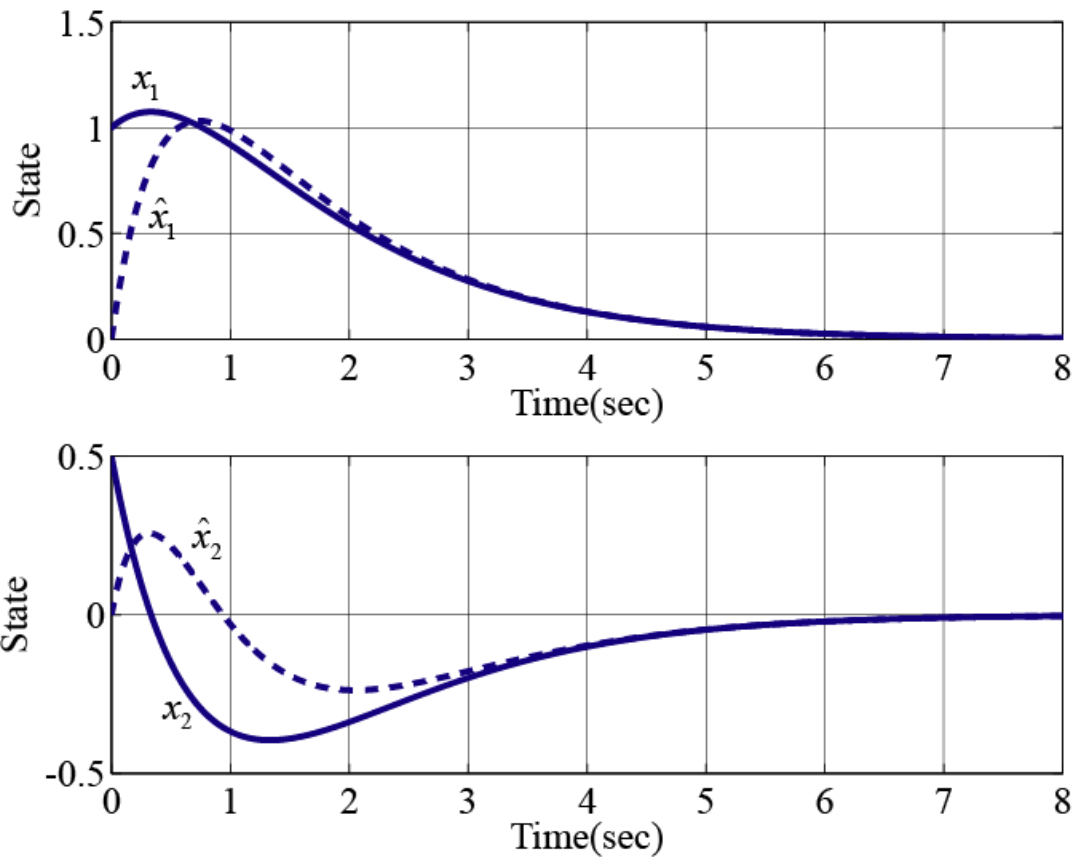
$$u = -Kx$$

$$K = [2 \quad 1]$$

Example



Example



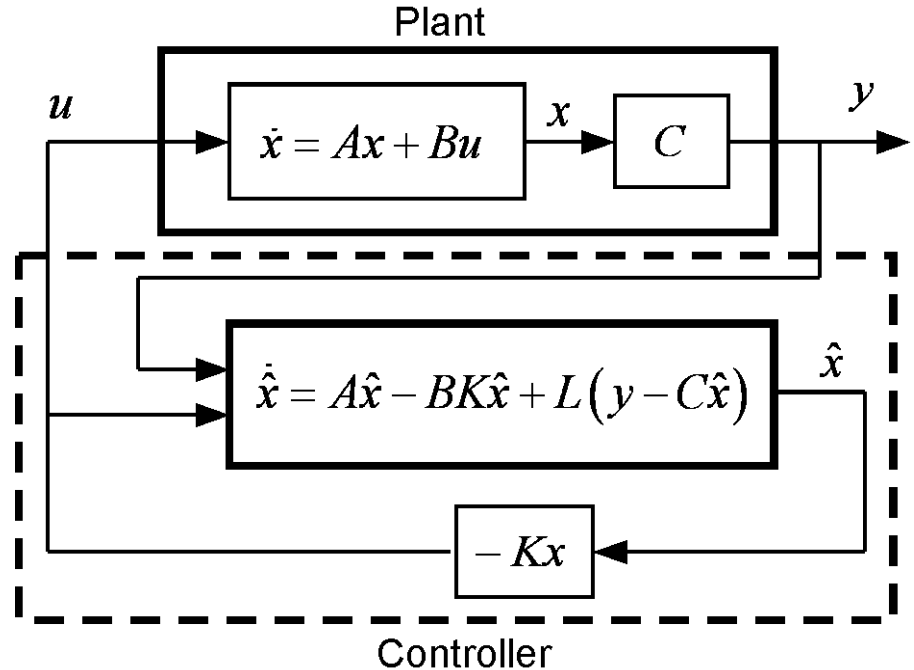
Estimator-Based Controller

- Zero reference input: regulator

$$u = -K\hat{x} \quad \dot{\hat{x}} = A\hat{x} - BK\hat{x} + L(y - C\hat{x})$$

$$\dot{x} = Ax - BK\hat{x}$$

$$\dot{\hat{x}} = LCx + (A - BK - LC)\hat{x}$$



Estimator-Based Controller

- Closed-loop system

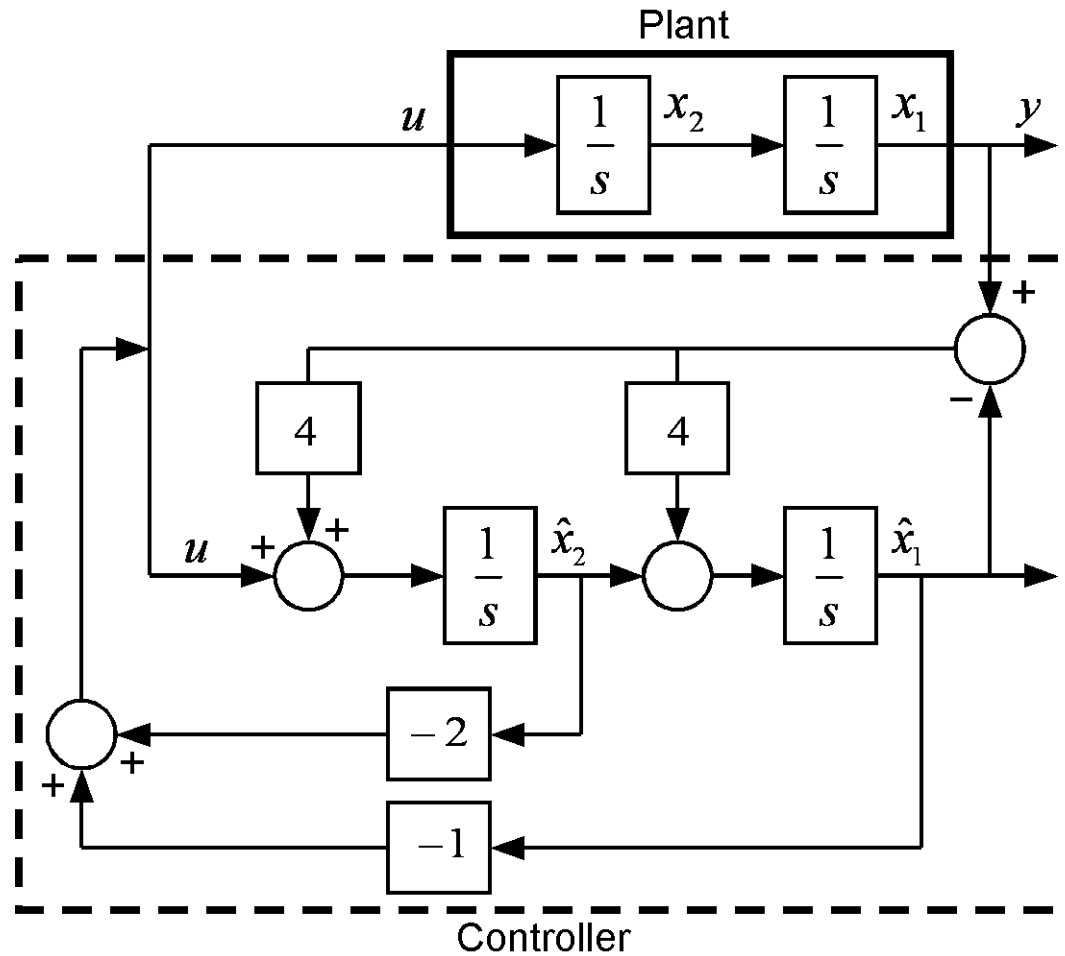
$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\hat{x}} = Ax - BK\hat{x} - LCx - (A - BK - LC)\hat{x} \\ &= (A - LC)(x - \hat{x}) = (A - LC)e\end{aligned}$$

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

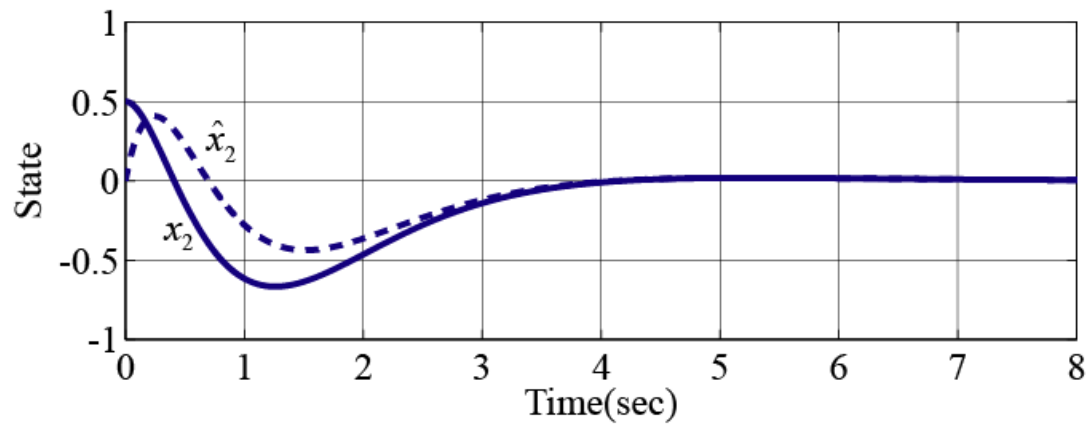
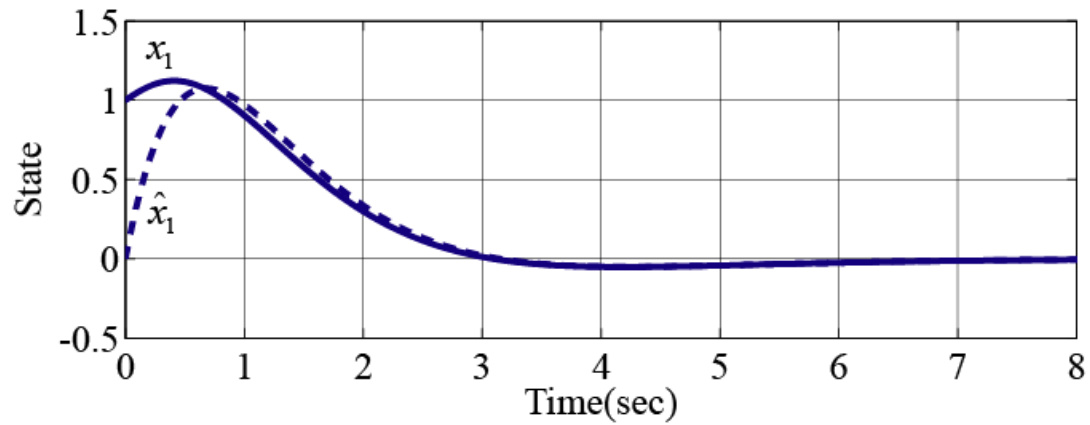
$$\left| \begin{bmatrix} sI & 0 \\ 0 & sI \end{bmatrix} - \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \right| = \left| \begin{matrix} sI - (A - BK) & -BK \\ 0 & sI - (A - LC) \end{matrix} \right| = 0$$

$$\left| sI - (A - BK) \right| \left| sI - (A - LC) \right| = 0$$

Example



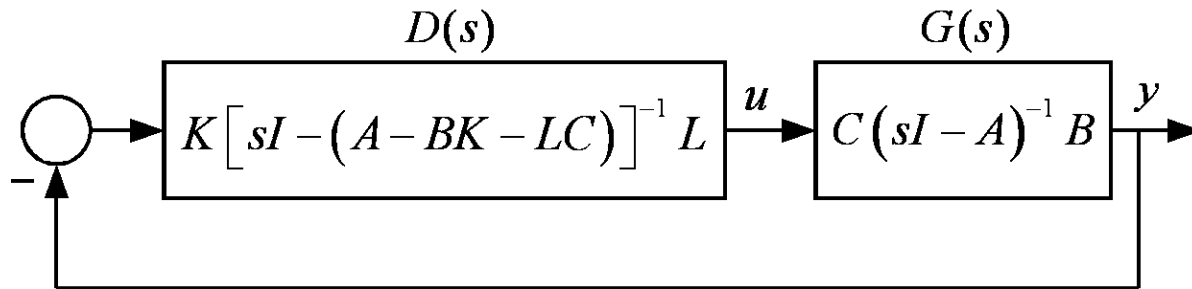
Example



Example

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly \quad u = -K\hat{x}$$

$$\frac{U(s)}{Y(s)} = -K \left[sI - (A - BK - LC) \right]^{-1} L$$

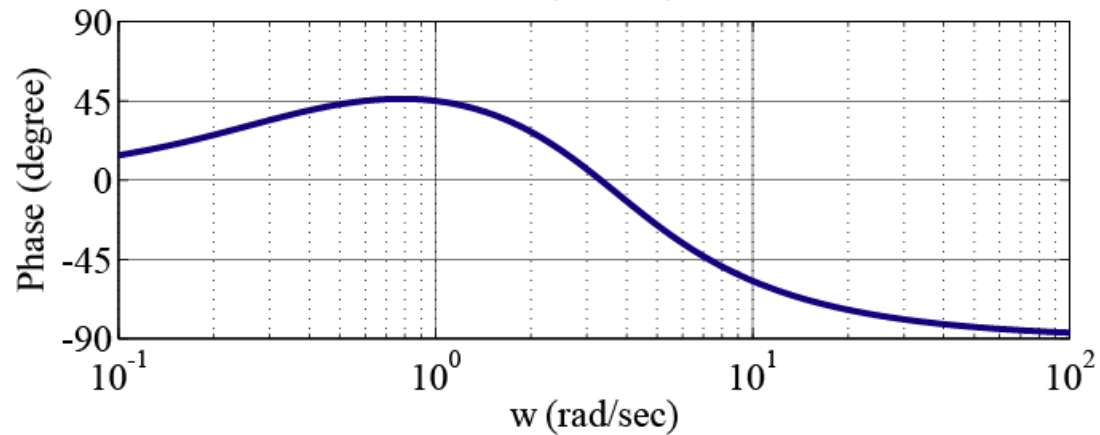
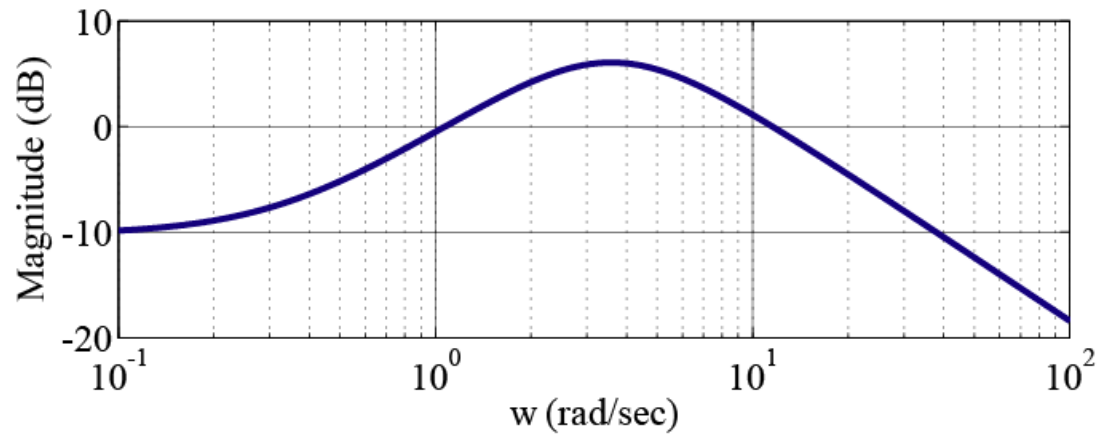


$$G(s) = C (sI - A)^{-1} B$$

$$D(s) = K \left[sI - (A - BK - LC) \right]^{-1} L$$

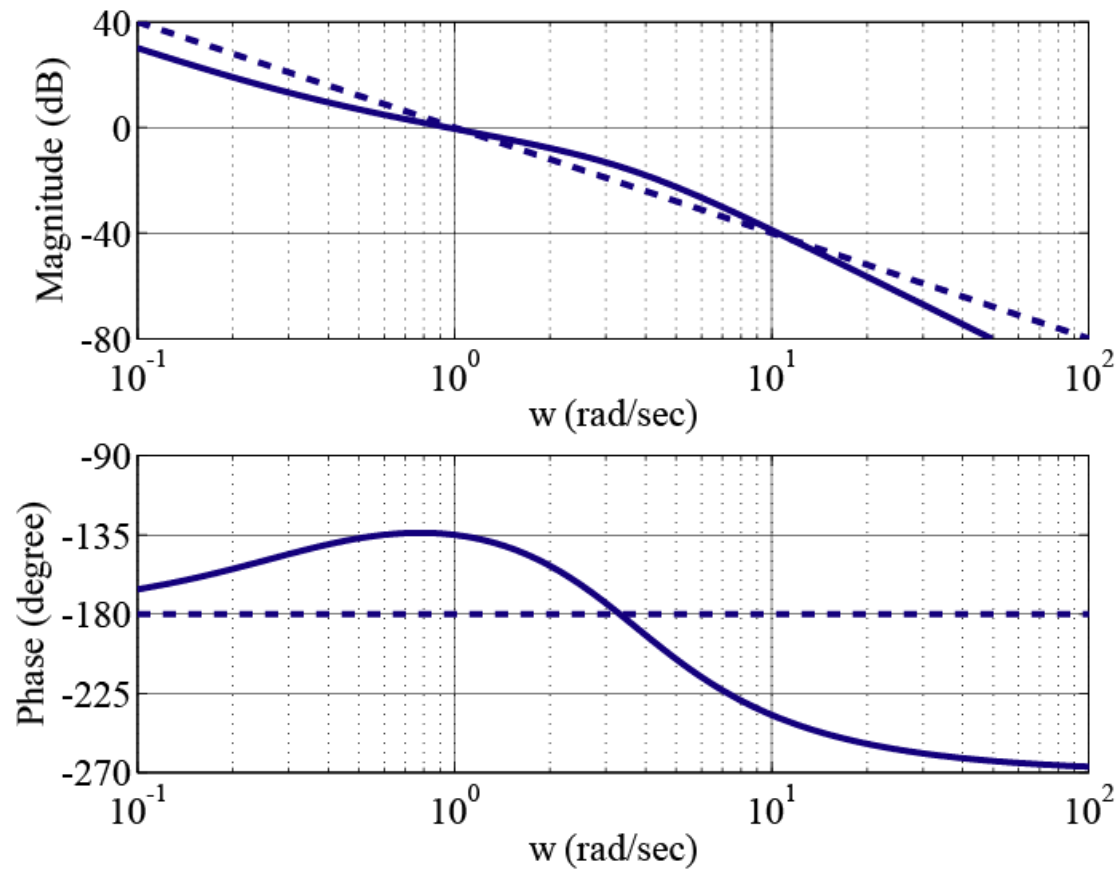
Example

$$D(s) = \frac{12s + 4}{s^2 + 6s + 13}$$



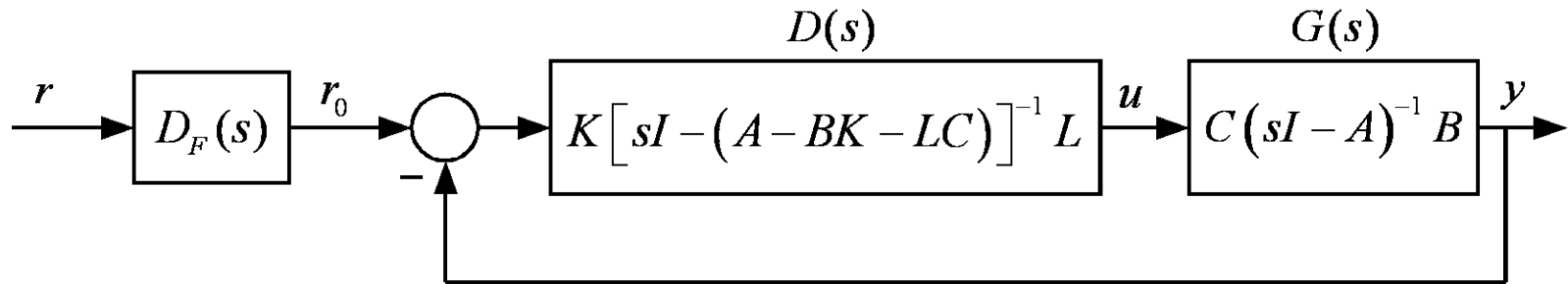
Example

- $D(s)G(s)$ Bode Plot



Estimator-Based Controller

- Non-zero reference input (1st Method)



$$\dot{\hat{x}} = (A - BK - LC) \hat{x} + L(y - r_0)$$

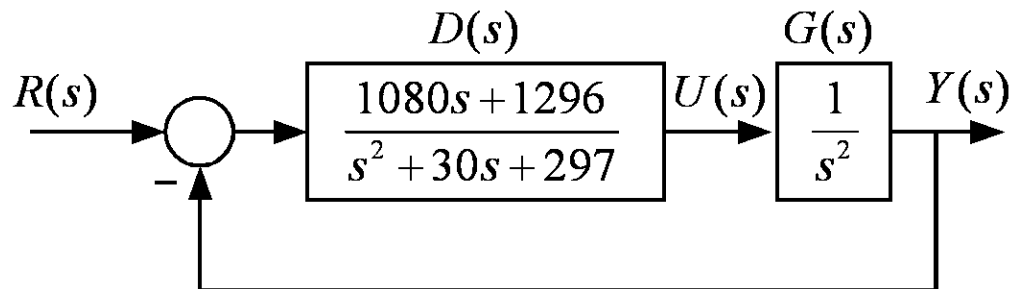
$$u = -K\hat{x}$$

Example

$$K = [9 \quad 6] \quad L = \begin{bmatrix} 24 \\ 144 \end{bmatrix}$$

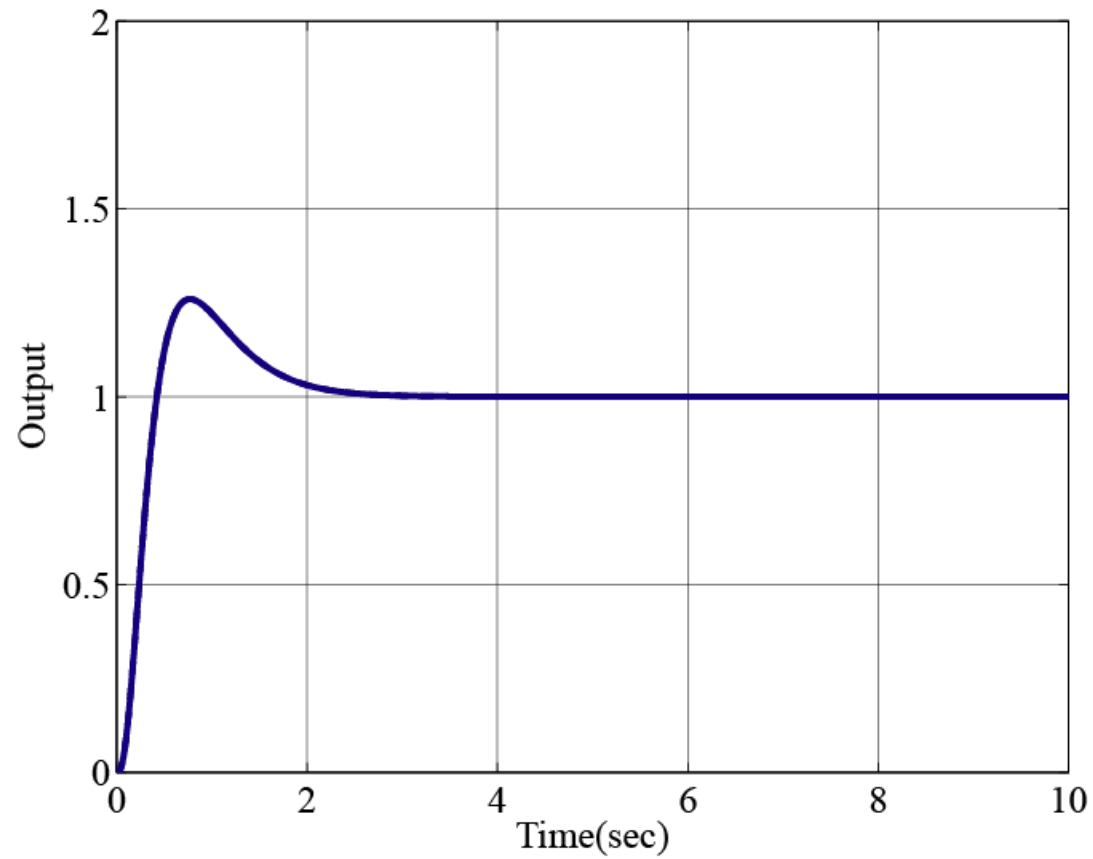
$$D(s) = K [sI - (A - BK - LC)]^{-1} L = \frac{1080s + 1296}{s^2 + 30s + 297}$$

$$G(s) = \frac{1}{s^2}$$



$$\frac{Y(s)}{R(s)} = \frac{1080s + 1296}{s^4 + 30s^3 + 297s^2 + 1080s + 1296}$$

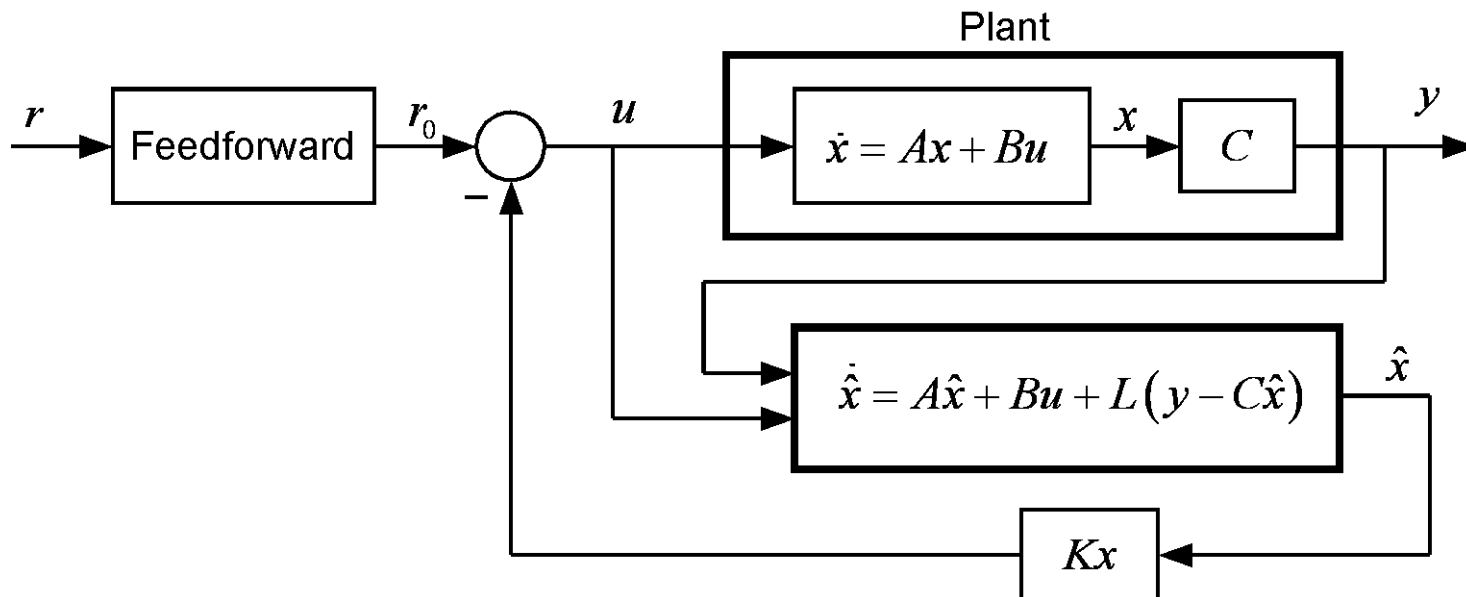
Example



Estimator-Based Controller

- Non-zero reference input (2nd method)

$$\dot{\hat{x}} = A\hat{x} + B(r_0 - K\hat{x}) + L(y - C\hat{x}) \quad u = r_0 - K\hat{x}$$



Estimator-Based Controller

- Closed-loop system

$$\dot{x} = Ax + B(r_0 - K\hat{x})$$

$$\dot{\hat{x}} = LCx + (A - LC)\hat{x} + B(r_0 - K\hat{x})$$

$$e = x - \hat{x}$$

$$\dot{x} = (A - BK)x + BKe + Br_0$$

$$\dot{e} = (A - LC)e$$

$$sX(s) - x(0) = (A - BK)X(s) + BKE(s) + BR_0(s)$$

$$sE(s) - e(0) = (A - LC)E(s)$$

Estimator-Based Controller

- Closed-loop system

$$X(s) = [sI - (A - BK)]^{-1} [x(0) + BKE(s) + BR_0(s)]$$

$$E(s) = [sI - (A - LC)]^{-1} e(0)$$

$$\frac{Y(s)}{R_0(s)} = C[sI - (A - BK)]^{-1} B$$

$$\frac{R_0(s)}{R(s)} = D_F(s)$$

Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 24 \\ 144 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 24 \\ 144 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

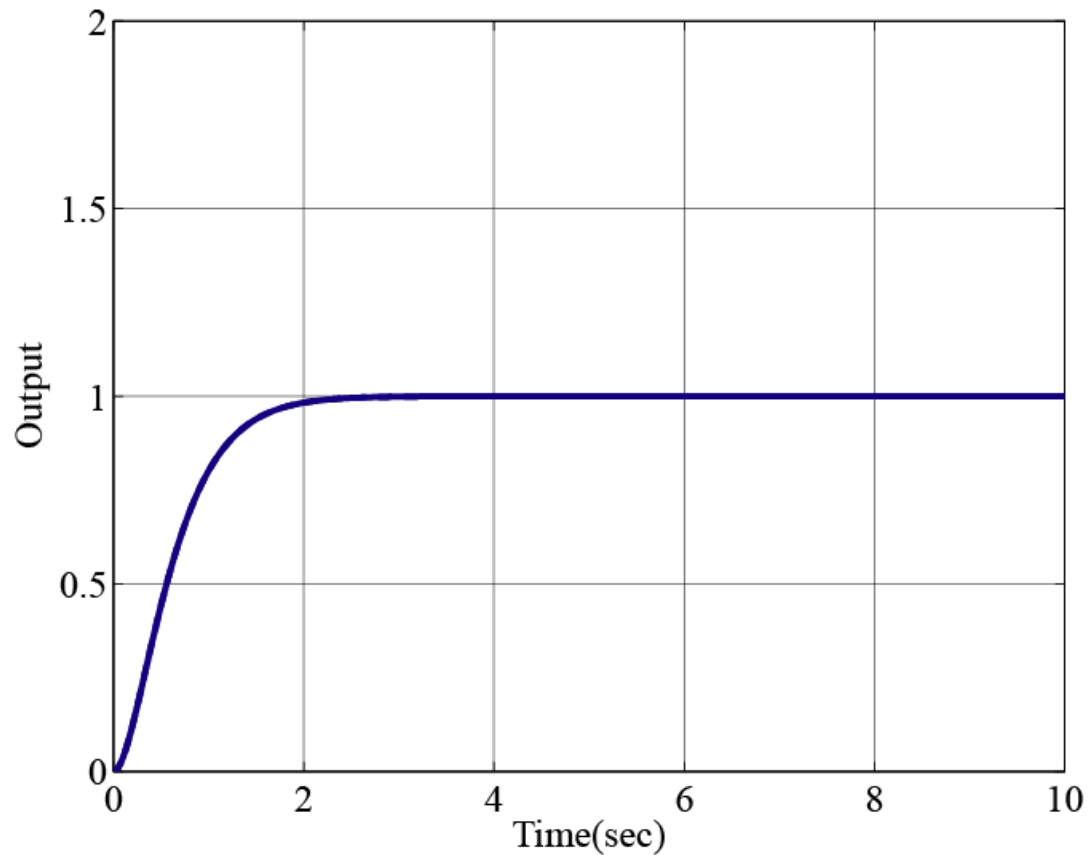
$$u = r_0 - \begin{bmatrix} 9 & 6 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{Y(s)}{R_0(s)} = \frac{(s+12)^2}{(s+3)^2(s+12)^2} = \frac{1}{s^2 + 6s + 9}$$

$$D_F(s) = 9$$

Example

$$\frac{Y(s)}{R(s)} = \frac{9}{s^2 + 6s + 9}$$



Example

$$D_F(s) = 6s + 9$$

$$\frac{Y(s)}{R(s)} = \frac{6s + 9}{s^2 + 6s + 9}$$

