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# **Discrete-Time State-Space Design**

# State Equation

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- Continuous-Time

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

- Discrete-Time

$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

# Example

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$$y(k+2) = u(k) + 1.7y(k+1) - 0.72y(k)$$

Let

$$x_1(k) = y(k)$$

$$x_2(k) = x_1(k+1) = y(k+1)$$

Then

$$x_1(k+1) = x_2(k)$$

$$x_2(k+1) = y(k+2) = -0.72x_1(k) + 1.7x_2(k) + u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.72 & 1.7 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

# Review: Continuous-Time Control Canonical Form

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$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

$$x_1(t) = y(t)$$

$$x_2(t) = y^{(1)}(t)$$

⋮

$$x_n(t) = y^{(n-1)}(t)$$

# Review: Continuous-Time Control Canonical Form

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$$\dot{x}_1(t) = y^{(1)}(t) = x_2(t)$$

$$\dot{x}_2(t) = y^{(2)}(t) = x_3(t)$$

⋮

$$\dot{x}_n(t) = y^{(n)}(t) = -a_0x_1(t) - a_1x_2(t) - \cdots - a_{n-1}x_n(t) + u(t)$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = [1 \ 0 \ 0 \ \cdots \ 0], D = 0$$

# Review: Continuous-Time Control Canonical Form

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$$\begin{aligned} \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) \\ = b_{n-1} \frac{d^{n-1} u(t)}{dt^{n-1}} + b_{n-2} \frac{d^{n-2} u(t)}{dt^{n-2}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t) \end{aligned}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} \cdot \frac{Z(s)}{Z(s)}$$

# Review: Continuous-Time Control Canonical Form

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$$U(s) = \left( s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 \right) Z(s)$$

$$Y(s) = \left( b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_1s + b_0 \right) Z(s)$$

$$\frac{d^n z(t)}{dt^n} + a_{n-1} \frac{d^{n-1} z(t)}{dt^{n-1}} + \dots + a_1 \frac{dz(t)}{dt} + a_0 z(t) = u(t)$$

$$x_1(t) = z(t)$$

$$x_2(t) = z^{(1)}(t)$$

⋮

$$x_n(t) = z^{(n-1)}(t)$$

# Review: Continuous-Time Control Canonical Form

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$$\dot{x}_1(t) = z^{(1)}(t) = x_2(t)$$

$$\dot{x}_2(t) = z^{(2)}(t) = x_3(t)$$

⋮

$$\dot{x}_n(t) = z^{(n)}(t) = -a_0 x_1(t) - a_1 x_2(t) - \cdots - a_{n-1} x_n(t) + u(t)$$

$$\begin{aligned} y(t) &= b_{n-1} \frac{d^{n-1} z(t)}{dt^{n-1}} + b_{n-2} \frac{d^{n-2} z(t)}{dt^{n-2}} + \cdots + b_1 \frac{dz(t)}{dt} + b_0 z(t) \\ &= b_{n-1} x_n(t) + b_{n-2} x_{n-1}(t) + \cdots + b_1 x_2(t) + b_0 x_1(t) \end{aligned}$$



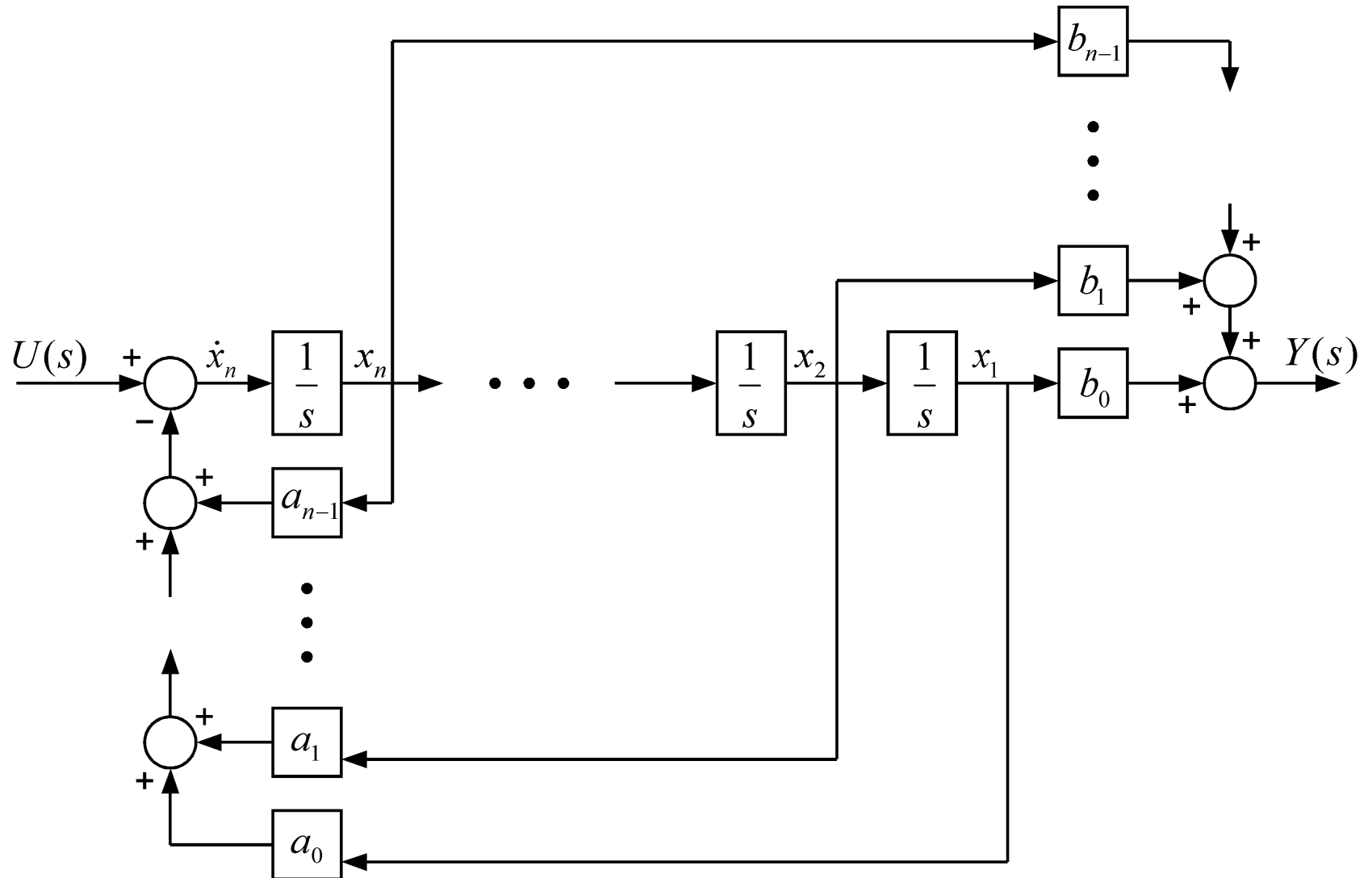
# Review: Continuous-Time Control Canonical Form

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$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

# Review: Continuous-Time Control Canonical Form



# Review: Continuous-Time Control Canonical Form

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$$G(s) = \frac{Y(s)}{U(s)} = \frac{5s + 4}{s^3 + 2s^2 + 3s + 1}$$

$$\frac{Y(s)}{U(s)} = \frac{5s + 4}{s^3 + 2s^2 + 3s + 1} \cdot \frac{Z(s)}{Z(s)}$$

$$U(s) = (s^3 + 2s^2 + 3s + 1)Z(s)$$

$$Y(s) = (5s + 4)Z(s)$$

$$x_1 = z, x_2 = \dot{z}, x_3 = \ddot{z}$$

# Review: Continuous-Time Control Canonical Form

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$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = \ddot{z} = -x_1 - 3x_2 - 2x_3 + u$$

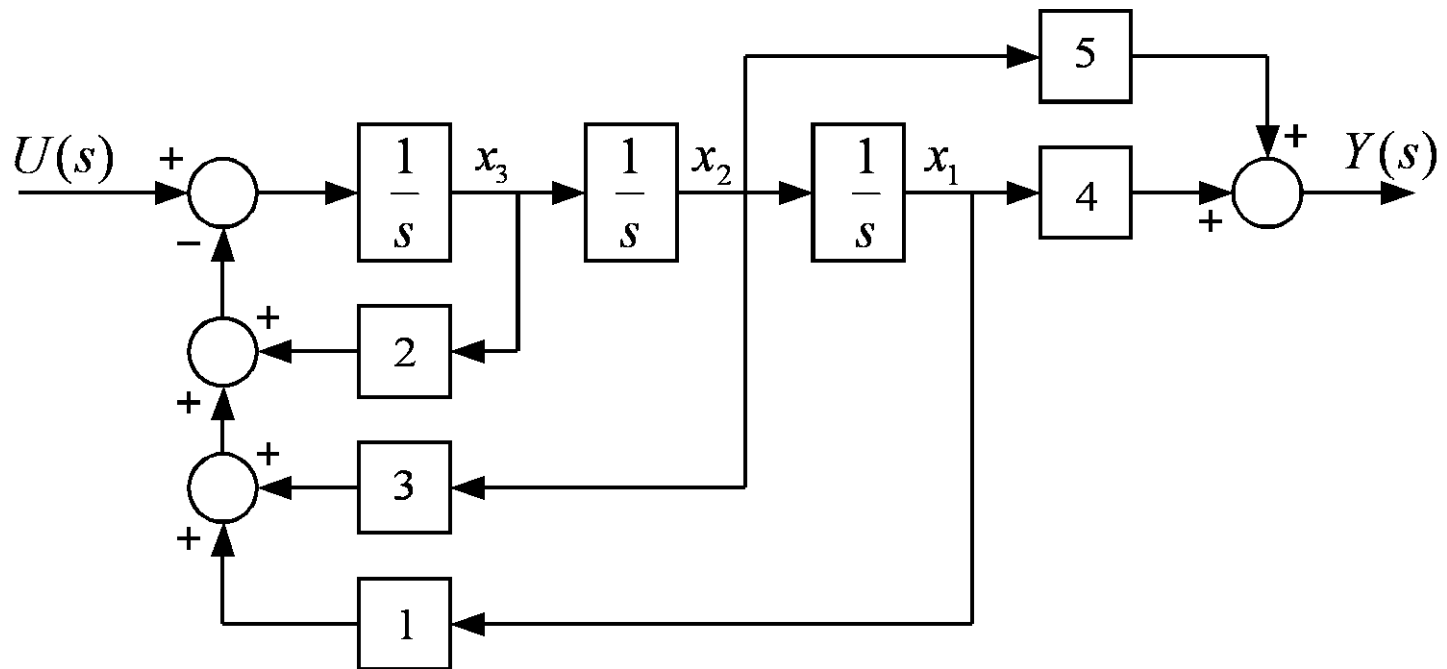
$$y = 4x_1 + 5x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

# Review: Continuous-Time Control Canonical Form

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# Discrete-Time Control Canonical Form

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$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1y(k+1) + a_0y(k) = u(k)$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0}$$

$$x_1(k) = y(k)$$

$$x_2(k) = y(k+1)$$

⋮

$$x_n(k) = y(k+n-1)$$

# Discrete-Time Control Canonical Form

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$$x_1(k+1) = y(k+1) = x_2(k)$$

$$x_2(k+1) = y(k+2) = x_3(k)$$

⋮

$$x_n(k+1) = y(k+n) = -a_0x_1(k) - a_1x_2(k) - \cdots - a_{n-1}x_n(k) + u(k)$$

$$y(k) = x_1(k)$$

# Discrete-Time Control Canonical Form

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$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0 \quad \cdots \quad 0 \quad 0] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$



# Discrete-Time Control Canonical Form

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$$\begin{aligned} y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1y(k+1) + a_0y(k) \\ = b_{n-1}u(k+n-1) + b_{n-2}u(k+n-2) + \cdots + b_1u(k+1) + b_0u(k) \end{aligned}$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0}$$

$$G(z) = \frac{Y(z)}{U(z)} = \frac{b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0} \cdot \frac{E(z)}{E(z)}$$

# Discrete-Time Control Canonical Form

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$$U(z) = \left( z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0 \right) E(z)$$

$$Y(z) = \left( b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \dots + b_1z + b_0 \right) E(z)$$

$$e(k+n) + a_{n-1}e(k+n-1) + \dots + a_1e(k+1) + a_0e(k) = u(k)$$

$$x_1(k) = e(k)$$

$$x_2(k) = e(k+1)$$

⋮

$$x_n(k) = e(k+n-1)$$

# Discrete-Time Control Canonical Form

---

$$x_1(k+1) = e(k+1) = x_2(k)$$

$$x_2(k+1) = e(k+2) = x_3(k)$$

⋮

$$x_n(k+1) = e(k+n) = -a_0x_1(k) - a_1x_2(k) - \cdots - a_{n-1}x_n(k) + u(k)$$

$$y(k) = b_{n-1}e(k+n-1) + b_{n-2}e(k+n-2) + \cdots + b_1e(k+1) + b_0e(k)$$

$$= b_{n-1}x_n(k) + b_{n-2}x_{n-1}(k) + \cdots + b_1x_2(k) + b_0x_1(k)$$

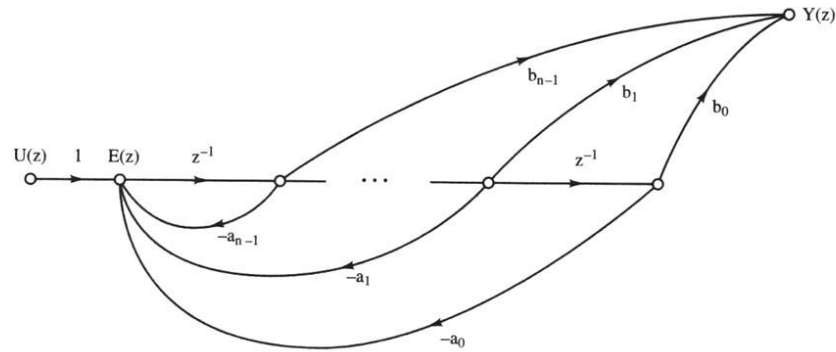
# Discrete-Time Control Canonical Form

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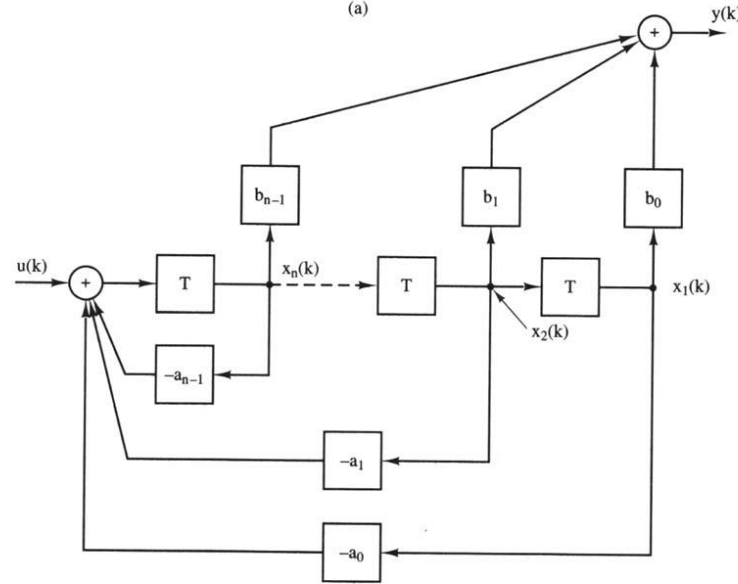
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k)$$

$$y(k) = [b_0 \quad b_1 \quad \cdots \quad b_{n-2} \quad b_{n-1}] \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

# Discrete-Time Control Canonical Form



(a)



(b)

**Figure 2-9** Equivalent representations of equation (2-51): (a) signal flow graph representation; (b) simulation diagram.

# Review: Continuous-Time System

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- 행렬 지수 함수(matrix exponential function)

$$\dot{x}(t) = Ax(t)$$

$$x(t) = e^{At} x(0)$$

$$e^{At} = I + At + \frac{A^2 t^2}{2} + \dots + \frac{A^n t^n}{n!} + \dots$$

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

# Review: Continuous-Time System

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$$\frac{de^{At}}{dt} = A + A^2t + \dots + \frac{A^n t^{n-1}}{(n-1)!} + \dots$$

$$= A \left( I + At + \frac{A^2 t^2}{2} + \dots \right) = Ae^{At} \quad \dot{x}(t) = Ae^{At} x(0) = Ax(t)$$

$$= \left( I + At + \frac{A^2 t^2}{2} + \dots \right) A = e^{At} A$$

$$sX(s) - x(0) = AX(s)$$

$$X(s) = (sI - A)^{-1} x(0)$$

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] x(0)$$

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]$$

$$\mathcal{L}\left[e^{At}\right] = (sI - A)^{-1}$$

# Review: Continuous-Time System

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$$\dot{x}(t) - Ax(t) = Bu(t)$$

$$e^{-At} [\dot{x}(t) - Ax(t)] = e^{-At} Bu(t)$$

$$\frac{d}{d} [e^{-At} x(t)] = e^{-At} \dot{x}(t) - Ae^{-At} x(t) = e^{-At} [\dot{x}(t) - Ax(t)]$$

$$\frac{d}{d} [e^{-At} x(t)] = e^{-At} Bu(t)$$

$$e^{-At} x(t) - x(0) = \int_0^t e^{-A\tau} Bu(\tau) d\tau$$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$



# Review: Continuous-Time System

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$$X(s) = \mathcal{L}\left[e^{At}\right]x(0) + \mathcal{L}\left[e^{At}\right]\mathcal{L}[Bu(t)]$$

$$\begin{aligned}x(t) &= \mathcal{L}^{-1}\left[X(s)\right] = e^{At}x(0) + \left[e^{At}\right] * [Bu(t)] \\ &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\end{aligned}$$

# Review: Continuous-Time System

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$$x(t_0) = e^{At_0} x(0) + \int_0^{t_0} e^{A(t_0-\tau)} Bu(\tau) d\tau$$

$$e^{At_0} e^{At_1} = e^{A(t_0+t_1)}$$

$$\begin{aligned} x(t) &= e^{A(t-t_0)} e^{At_0} x(0) + \int_0^{t_0} e^{A(t-t_0)} e^{A(t_0-\tau)} Bu(\tau) d\tau + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau \\ &= e^{A(t-t_0)} \left( e^{At_0} x(0) + \int_0^{t_0} e^{A(t_0-\tau)} Bu(\tau) d\tau \right) + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau \end{aligned}$$

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau$$

$$y(t) = Cx(t) + Du(t) = Ce^{A(t-t_0)} x(t_0) + C \int_{t_0}^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

# Discrete-Time State Equations

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$$\dot{v}(t) = A_c v(t) + B_c u(t)$$

$$y(t) = C_c v(t) + D_c u(t)$$

$$v(t) = e^{A_c(t-t_0)} v(t_0) + \int_{t_0}^t e^{A_c(t-\tau)} B_c u(\tau) d\tau$$

$$t = kT + T, t_0 = kT$$

$$v(kT + T) = e^{A_c T} v(kT) + u(kT) \int_{kT}^{kT+T} e^{A_c(kT+T-\tau)} B_c d\tau$$

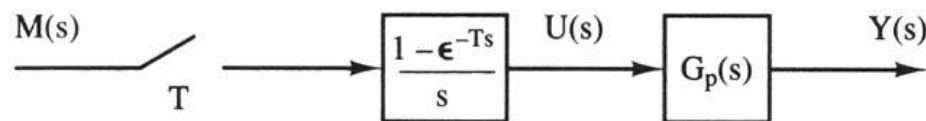


Figure 4-19 Sampled-data system.

# Discrete-Time State Equations

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$$x(k+1) = Ax(k) + Bu(k)$$

$$y(k) = Cx(k) + Du(k)$$

$$x(k) = v(kT)$$

$$A = e^{A_c T}$$

$$B = \int_{kT}^{kT+T} e^{A_c(kT+T-\tau)} B_c d\tau = \left( \int_0^T e^{A_c \tau} d\tau \right) B_c$$

# Review: Continuous-Time System

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$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

# Review: Continuous-Time System

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$$X(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} BU(s)$$

$$Y(s) = C(sI - A)^{-1} x(0) + \left[ C(sI - A)^{-1} B + D \right] U(s)$$

$$Y(s) = \left[ C(sI - A)^{-1} B + D \right] U(s)$$

$$G(s) = C(sI - A)^{-1} B + D$$

$$Y(s) = G(s)U(s)$$

# Review: Continuous-Time System Example

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$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] x$$

$$\left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$G(s) = \frac{1}{s^2 + 3s + 2} [1 \quad 0] \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + 3s + 2}$$

# Review: Continuous-Time System Example

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$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] x + 4u$$

$$\begin{aligned} G(s) &= \frac{1}{s^2 + 3s + 2} [1 \quad 0] \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 4 = \frac{1}{s^2 + 3s + 2} + 4 \\ &= \frac{4s^2 + 12s + 9}{s^2 + 3s + 2} \end{aligned}$$



# Review: Continuous-Time System

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- Poles and eigenvalues

$$|sI - A| = 0$$

$$Ax = sx \quad (x \neq 0)$$

$$(sI - A)x = 0$$

# Discrete State Equation: Transfer Function

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$$zX(z) = AX(z) + BU(z)$$

$$Y(z) = CX(z) + DU(z)$$

$$(zI - A)X(z) = BU(z)$$

$$X(z) = (zI - A)^{-1} BU(z)$$

$$Y(z) = C(zI - A)^{-1} BU(z) + DU(z)$$

$$G(z) = C(zI - A)^{-1} B + D$$

# Discrete State Equation: Poles

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$$\left| (zI - A) \right| = 0$$

# State Feedback Controller

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$$x(k+1) = Ax(k) + Bu(k)$$

$$u = -Kx(k)$$

$$x(k+1) = Ax(k) - BKx(k) = (A - BK)x(k)$$

$$\left| zI - (A - BK) \right| = 0$$

# Discrete-Time State Estimator

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$$\hat{x}(k+1) = A_e \hat{x}(k) + B_e u(k) + Ly(k)$$

$$\hat{x}(k) \rightarrow x(k), k \rightarrow \infty$$

$$e = x - \hat{x} \quad e \rightarrow 0, k \rightarrow \infty$$

$$\begin{aligned} x(k+1) - \hat{x}(k+1) &= Ax(k) + Bu(k) - A_e \hat{x}(k) - B_e u(k) - Ly(k) \\ &= Ax(k) + Bu(k) - A_e \hat{x}(k) - B_e u(k) - LCx(k) \\ &= (A - LC)x(k) - A_e \hat{x}(k) + (B - B_e)u(k) \end{aligned}$$

$$A_e = A - LC \quad B_e = B \quad e(k+1) = (A - LC)e(k)$$

$$\begin{aligned} \hat{x}(k+1) &= (A - LC)\hat{x}(k) + Bu(k) + Ly(k) \\ &= A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k)) \end{aligned}$$

# Estimator-Based Controller

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$$\hat{x}(k+1) = A\hat{x}(k) - BK\hat{x}(k) + L(y(k) - C\hat{x}(k))$$

$$u(k) = -K\hat{x}(k)$$

- With a non-zero reference step

$$\hat{x}(k+1) = A\hat{x}(k) + B(K_1 ref - K\hat{x}(k)) + L(y(k) - C\hat{x}(k))$$

$$u(k) = K_1 ref - K\hat{x}(k)$$

# Closed-Loop

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- With a non-zero reference step

$$x(k+1) = Ax(k) + Bu(k) = Ax(k) + B(K_1 ref - K\hat{x}(k))$$

$$\hat{x}(k+1) = A\hat{x}(k) + B(K_1 ref - K\hat{x}(k)) + L(y(k) - C\hat{x}(k))$$

$$\begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - BK - LC \end{bmatrix} \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} K_1 ref$$

```
Ts=1/100; % sampling period  
Gc=ss([0 10;0 -10],[0;10],[1 0],0)  
Gd=c2d(Gc,Ts,'zoh')  
[A,B,C,D,Ts]=ssdata(Gd)  
pc1=-10+12*j,pc2=-10-12*j  
p=[exp(Ts*pc1) exp(Ts*pc2)]  
pe=[exp(Ts*pc1*5) exp(Ts*pc2*5)]  
  
K=acker(A,B,p)  
G=acker(A',C',pe)'
```