

Chapter 12

Feedback Control

Stabilization

We want to stabilize the system

$$\dot{x} = f(x, u)$$

at the equilibrium point $x = x_{ss}$

Steady-State Problem: Find steady-state control u_{ss} s.t.

$$0 = f(x_{ss}, u_{ss})$$

$$x_\delta = x - x_{ss}, \quad u_\delta = u - u_{ss}$$

$$\dot{x}_\delta = f(x_{ss} + x_\delta, u_{ss} + u_\delta) \stackrel{\text{def}}{=} f_\delta(x_\delta, u_\delta)$$

$$f_\delta(0, 0) = 0$$

$$u_\delta = \gamma(x_\delta) \Rightarrow u = u_{ss} + \gamma(x - x_{ss})$$

State Feedback Stabilization: Given

$$\dot{x} = f(x, u) \quad [f(0, 0) = 0]$$

find

$$u = \gamma(x) \quad [\gamma(0) = 0]$$

s.t. the origin is an asymptotically stable equilibrium point of

$$\dot{x} = f(x, \gamma(x))$$

f and γ are locally Lipschitz functions

Linear Systems

$$\dot{x} = Ax + Bu$$

(A, B) is stabilizable (controllable or every uncontrollable eigenvalue has a negative real part)

Find K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Typical methods:

- Eigenvalue Placement
- Eigenvalue-Eigenvector Placement
- LQR

Linearization

$$\dot{x} = f(x, u)$$

$f(0, 0) = 0$ and f is continuously differentiable in a domain $D_x \times D_u$ that contains the origin ($x = 0, u = 0$)
($D_x \subset \mathbb{R}^n, D_u \subset \mathbb{R}^p$)

$$\dot{x} = Ax + Bu$$

$$A = \left. \frac{\partial f}{\partial x}(x, u) \right|_{x=0, u=0} ; \quad B = \left. \frac{\partial f}{\partial u}(x, u) \right|_{x=0, u=0}$$

Assume (A, B) is stabilizable. Design a matrix K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Closed-loop system:

$$\dot{x} = f(x, -Kx)$$

$$\begin{aligned}\dot{x} &= \left[\frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx) (-K) \right]_{x=0} x \\ &= (A - BK)x\end{aligned}$$

Since $(A - BK)$ is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system

Example (Pendulum Equation):

$$\ddot{\theta} = -a \sin \theta - b\dot{\theta} + cT$$

Stabilize the pendulum at $\theta = \delta$

$$0 = -a \sin \delta + cT_{ss}$$

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{ss}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a[\sin(x_1 + \delta) - \sin \delta] - bx_2 + cu$$

$$A = \begin{bmatrix} 0 & 1 \\ -a \cos(x_1 + \delta) & -b \end{bmatrix}_{x_1=0} = \begin{bmatrix} 0 & 1 \\ -a \cos \delta & -b \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -a \cos \delta & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -(a \cos \delta + ck_1) & -(b + ck_2) \end{bmatrix}$$

$$k_1 > -\frac{a \cos \delta}{c}, \quad k_2 > -\frac{b}{c}$$

$$T = \frac{a \sin \delta}{c} - Kx = \frac{a \sin \delta}{c} - k_1(\theta - \delta) - k_2\dot{\theta}$$

Notions of Stabilization

$$\dot{x} = f(x, u), \quad u = \gamma(x)$$

Local Stabilization: The origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable and a given region G is a subset of the region of attraction (for all $x(0) \in G$, $\lim_{t \rightarrow \infty} x(t) = 0$) (e.g., $G \subset \Omega_c = \{V(x) \leq c\}$ where Ω_c is an estimate of the region of attraction)

Global Stabilization: The origin of $\dot{x} = f(x, \gamma(x))$ is globally asymptotically stable

Semiglobal Stabilization: The origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable and $\gamma(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \gamma_p(x)$ is dependent on a parameter p such that for any compact set G , p can be chosen to ensure that G is a subset of the region of attraction)

What is the difference between global stabilization and semiglobal stabilization?

Example

$$\dot{x} = x^2 + u$$

Linearization:

$$\dot{x} = u, \quad u = -kx, \quad k > 0$$

Closed-loop system:

$$\dot{x} = -kx + x^2$$

Linearization of the closed-loop system yields $\dot{x} = -kx$. Thus, $u = -kx$ achieves local stabilization

The region of attraction is $\{x < k\}$. Thus, for any set $\{x \leq a\}$ with $a < k$, the control $u = -kx$ achieves regional stabilization

The control $u = -kx$ does not achieve global stabilization

But it achieves semiglobal stabilization because any compact set $\{|x| \leq r\}$ can be included in the region of attraction by choosing $k > r$

The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable