Chapter 13
Feedback Linearization

Input-Output Linearization

Input-State Linearization

Stabilizing of the origin of the pendulum

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a[\sin(x_1+\delta)-\sin\delta]-bx_2+cu\end{aligned}$$

$$u = \frac{a}{c}[\sin(x_1+\delta)-\sin\delta]+\frac{v}{c}$$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -bx_2+v\end{aligned}$$

$$v = -k_1x_1-k_2x_2$$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -k_1x_1-(k_2+b)x_2\end{aligned}$$

$$u = \left(\frac{a}{c}\right)[\sin(x_1+\delta)-\sin\delta]-\frac{1}{c}(k_1x_1+k_2x_2)$$

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)] \tag{13.1}$$

where A is $n \times n$, B is $n \times p$, the pair (A, B) is controllable, the functions $\alpha : R^n \to R^p$ and $\gamma : R^n \to R^{p \times p}$ are defined in a domain $D \subset R^n$ that contains the origin, and the matrix $\gamma(x)$ is nonsingular for every $x \in D$. If the state equation takes the form (13.1), then we can linearize it via the state feedback

$$u = \alpha(x) + \beta(x)v \tag{13.2}$$

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)] \tag{13.1}$$

where A is $n \times n$, B is $n \times p$, the pair (A, B) is controllable, the functions $\alpha : R^n \to R^p$ and $\gamma : R^n \to R^{p \times p}$ are defined in a domain $D \subset R^n$ that contains the origin, and the matrix $\gamma(x)$ is nonsingular for every $x \in D$. If the state equation takes the form (13.1), then we can linearize it via the state feedback

$$u = \alpha(x) + \beta(x)v \tag{13.2}$$

$$\begin{aligned} \dot{x}_1 &= a \sin x_2 \\ \dot{x}_2 &= -x_1^2 + u \end{aligned}$$

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= a \sin x_2 = \dot{x}_1 \end{aligned}$$

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= a \cos x_2 \, (-x_1^2 + u) \end{aligned}$$

$$u = x_1^2 + \frac{1}{a \cos x_2} v \qquad -\pi/2 < x_2 < \pi/2$$

$$
\begin{aligned}
x_1 &= z_1 \\
x_2 &= \sin^{-1}\left(\frac{z_2}{a}\right) \qquad\qquad -a < z_2 < a
\end{aligned}
$$

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= a\cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)(-z_1^2 + u)
\end{aligned}
$$

Input-State Linearization

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u)$$

$$\dot{x}_1 = -2x_1 + a\,x_2 + \sin x_1 \qquad z_1 = x_1$$

$$\dot{x}_2 = -x_2 \cos x_1 + u \cos(2x_1) \qquad z_2 = a x_2 + \sin x_1$$

$$\dot{z}_1 = -2z_1 + z_2$$

$$\dot{z}_2 = -2z_1 \cos z_1 + \cos z_1 \sin z_1 + a u \cos(2z_1)$$

$$u = \frac{1}{a\cos(2z_1)}(v - \cos z_1 \sin z_1 + 2z_1 \cos z_1)$$

Input-State Linearization

$$\dot{z}_1 = -2z_1 + z_2$$

$$\dot{z}_2 = v$$

$$v = -k_1 z_1 - k_2 z_2$$

may choose

$$v = -2z_2 \tag{6.16}$$

resulting in the stable closed-loop dynamics

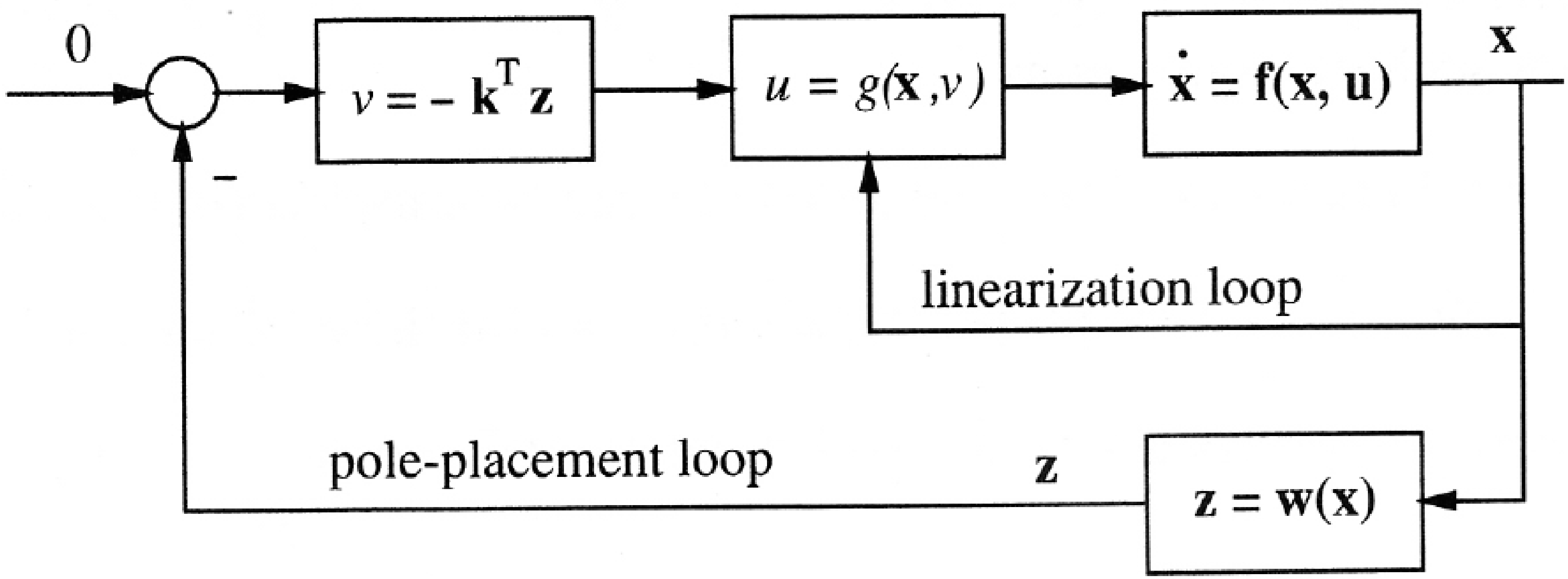
$$\dot{z}_1 = -2z_1 + z_2$$

$$\dot{z}_2 = -2z_2$$

whose poles are both placed at -2. In terms of the original state x_1 and x_2, this control law corresponds to the original input

$$u = \frac{1}{\cos(2x_1)}\left(-2ax_2 - 2\sin x_1 - \cos x_1 \sin x_1 + 2x_1 \cos x_1\right) \tag{6.17}$$

Input-State Linearization



Input-State Linearization

- The result, though valid in a large region of the state space, is not global. The control law is not well defined when $x_1 = (\pi/4 \pm k\pi/2)$, $k = 1, 2, ...$ Obviously, when the initial state is at such singularity points, the controller cannot bring the system to the equilibrium point.

- The input-state linearization is achieved by a combination of a state transformation and an input transformation, with state feedback used in both. Thus, it is a linearization by feedback, or feedback linearization. This is fundamentally different from a Jacobian linearization for small range operation, on which linear control is based.

- In order to implement the control law, the new state components (z_1, z_2) must be available. If they are not physically meaningful or cannot be measured directly, the original state $\mathbf{x}$ must be measured and used to compute them from (6.12).

Input-State Linearization

- What classes of nonlinear systems can be transformed into linear systems?
- How to find the proper transformations for those which can?

When a change of variables $z = T(x)$ is used to transform the state equation from the x-coordinates to the z-coordinates, the map T must be invertible; that is, it must have an inverse map $T^{-1}(\cdot)$ such that $x = T^{-1}(z)$ for all $z \in T(D)$, where D is the domain of T. Moreover, because the derivatives of z and x should be continuous, we require both $T(\cdot)$ and $T^{-1}(\cdot)$ to be continuously differentiable. A continuously differentiable map with a continuously differentiable inverse is known as a *diffeomorphism*. If the Jacobian matrix $[\partial T/\partial x]$ is nonsingular at a point $x_0 \in D$, then it follows from the inverse function theorem[1] that there is a neighborhood N of x_0 such that T restricted to N is a diffeomorphism on N. A map T is said to be a global diffeomorphism if it is a diffeomorphism on R^n and $T(R^n) = R^n$.[2] Now we have all the elements we need to define feedback linearizable systems.

Definition 13.1 *A nonlinear system*

$$\dot{x} = f(x) + G(x)u \tag{13.5}$$

where $f : D \rightarrow R^n$ *and* $G : D \rightarrow R^{n\times p}$ *are sufficiently smooth*[3] *on a domain* $D \subset R^n$, *is said to be feedback linearizable (or input–state linearizable) if there exists a diffeomorphism* $T : D \rightarrow R^n$ *such that* $D_z = T(D)$ *contains the origin and the change of variables* $z = T(x)$ *transforms the system* (13.5) *into the form*

$$\dot{z} = Az + B\gamma(x)[u - \alpha(x)] \tag{13.6}$$

with (A, B) *controllable and* $\gamma(x)$ *nonsingular for all* $x \in D$.

Input-Output Linearization

Let us now consider a tracking control problem. Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u) \tag{6.19a}$$

$$y = h(\mathbf{x}) \tag{6.19b}$$

and assume that our objective is to make the output $y(t)$ track a desired trajectory $y_d(t)$ while keeping the whole state bounded, where $y_d(t)$ and its time derivatives up to a sufficiently high order are assumed to be known and bounded. An apparent difficulty with this model is that the output y is only indirectly related to the input u, through the state variable $\mathbf{x}$ and the nonlinear state equations (6.19). Therefore, it is not easy to see how the input u can be designed to control the tracking behavior of the output y. However, inspired by the results of section 6.1.1, one might guess that the difficulty of the tracking control design can be reduced if we can *find a direct and simple relation between the system output y and the control input u.* Indeed, this idea constitutes the intuitive basis for the so-called input-output linearization approach to nonlinear control design. Let us again use an example to demonstrate this approach.

Input-Output Linearization

$$\dot{x}_1 = \sin x_2 + (x_2 + 1)\, x_3$$

$$\dot{x}_2 = x_1{}^5 + x_3$$

$$\dot{x}_3 = x_1{}^2 + u$$

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = \sin x_2 + (x_2 + 1)\, x_3$$

$$\ddot{y} = (x_2 + 1)\, u + f_1(\mathbf{x})$$

$$f_1(\mathbf{x}) = (x_1{}^5 + x_3)\,(x_3 + \cos x_2) + (x_2 + 1)\, x_1{}^2$$

Input-Output Linearization

$$u = \frac{1}{x_2 + 1}(v - f_1)$$

$$\ddot{y} = v \qquad e = y(t) - y_d(t)$$

$$v = \ddot{y}_d - k_1 e - k_2 \dot{e} \qquad \ddot{e} + k_2 \dot{e} + k_1 e = 0$$

seen in section 6.4 for SISO systems and in section 6.5 for MIMO systems. If we need to differentiate the output of a system r times to generate an explicit relationship between the output y and input u, the system is said to have *relative degree* r. Thus, the system in the above example has relative degree 2. As will be shown soon, this terminology is consistent with the notion of relative degree in linear systems (excess of poles over zeros). As we shall see later, it can also be shown formally that for any

Internal Dynamics

Therefore, a part of the system dynamics (described by one state component) has been rendered "unobservable" in the input-output linearization. This part of the dynamics will be called the *internal dynamics*, because it cannot be seen from the external input-output relationship (6.21). For the above example, the internal state can be chosen to be x_3 (because x_3, and y and $\dot{y}$, constitute a new set of states), and the internal dynamics is represented by the equation

$$\dot{x}_3 = {x_1}^2 + \frac{1}{x_2 + 1}\left(\ddot{y}_d(t) - k_1 e - k_2 \dot{e} + f_1 \right) \tag{6.26}$$

Example: Internal Dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} {x_2}^3 + u \\ u \end{bmatrix}$$

$$y = x_1$$

$$u = -{x_2}^3 - e(t) + \dot{y}_d(t) \qquad \dot{e} + e = 0$$

$$\dot{x}_2 + {x_2}^3 = \dot{y}_d - e \qquad | \dot{y}_d(t) - e | \le D$$

Internal Dynamics of Linear Systems

- Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ u \end{bmatrix}$$

$$y = x_1$$

$$\dot{y} = x_2 + u$$

$$u = -x_2 + \dot{y}_d - (y - y_d)$$

$$\dot{e} + e = 0$$

$$\dot{x}_2 + x_2 = \dot{y}_d - e(t)$$

Internal Dynamics of Linear Systems

- Example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u \\ -u \end{bmatrix}$$

$$\dot{x}_2 - x_2 = e(t) - \dot{y}_d$$

$$y = x_1$$

$$W_1(p) = \frac{p+1}{p^2} \qquad W_2(p) = \frac{p-1}{p^2}$$

Internal Dynamics of Linear Systems

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}u \qquad y = \mathbf{c}^T\mathbf{z}$$

$$y = \mathbf{c}^T(p\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}\,u = \frac{b_o + b_1 p}{a_o + a_1 p + a_2 p^2 + p^3}\,u$$

$$x_1 = \frac{1}{a_o + a_1 p + a_2 p^2 + p^3}\,u$$

$$x_2 = \dot{x}_1$$

$$x_3 = \dot{x}_2$$

Internal Dynamics of Linear Systems

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [b_o \quad b_1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\dot{y} = b_o x_2 + b_1 x_3$$

$$\ddot{y} = b_o \dot{x}_2 + b_1 \dot{x}_3 = b_o x_3 + b_1(-a_o x_1 - a_1 x_2 - a_2 x_3 + u)$$

Internal Dynamics of Linear Systems

$$u = (a_o x_1 + a_1 x_2 + a_2 x_3 - \frac{b_o}{b_1} x_3) + \frac{1}{b_1}(-k_1 e - k_2 \dot{e} + \ddot{y}_d)$$

$$e = y - y_d \qquad \ddot{e} + k_2 \dot{e} + k_1 e = 0$$

the state vector, since one can easily show x_1, y, and $\dot{y}$ are related to x_1, x_2, and x_3 through a one-to-one transformation (and thus can serve as states for the system). We then easily find from (6.36a) and (6.36b) that the internal dynamics is

$$\dot{x}_1 = x_2 = \frac{1}{b_1}(y - b_o x_1)$$

that is,

$$\dot{x}_1 + \frac{b_o}{b_1} x_1 = \frac{1}{b_1} y \tag{6.40}$$

Zero Dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^3 + u \\ u \end{bmatrix}$$

$$y = x_1$$

$$u = -x_2^3 - e(t) + \dot{y}_d(t) \qquad \dot{e} + e = 0$$

$$\dot{x}_2 + x_2^3 = \dot{y}_d - e \qquad |\,\dot{y}_d(t) - e\,| \leq D$$

A way to approach these difficulties is to define a so-called *zero-dynamics* for a nonlinear system. *The zero-dynamics is defined to be the internal dynamics of the system when the system output is kept at zero by the input.* For instance, for the system (6.27), the zero-dynamics is (from (6.30))

$$\dot{x}_2 + x_2^3 = 0 \tag{6.41}$$

Control Design Based on Input-Output Linearization

- *differentiate the output y until the input u appears*
- *choose u to cancel the nonlinearities and guarantee tracking convergence*
- *study the stability of the internal dynamics*

Input-Output Linearization

$$\begin{aligned} \dot{x}_1 &= a \sin x_2 \\ \dot{x}_2 &= -x_1^2 + u \end{aligned} \qquad y = x_2$$

$$z_1 = x_1, \qquad z_2 = a \sin x_2, \qquad \text{and} \quad u = x_1^2 + \frac{1}{a \cos x_2} v$$

yield

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= v \\ y &= \sin^{-1}\left(\frac{z_2}{a}\right) \end{aligned}$$

Input-Output Linearization

$$u = x_1^2 + v \qquad \begin{aligned} \dot{x}_2 &= v \\ y &= x_2 \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= a \sin x_2 \\ \dot{x}_2 &= v \\ y &= x_2 \end{aligned}$$

Note that the state variable x_1 is not connected to the output y. In other words, the linearizing feedback control has made x_1 unobservable from y. When we design tracking control, we should make sure that the variable x_1 is well behaved; that is, stable or bounded in some sense. A naive control design that uses only the linear input–output map may result in an ever-growing signal $x_1(t)$. For example, suppose we design a linear control to stabilize the output y at a constant value r. Then, $x_1(t) = x_1(0) + t\, a \sin r$ and, for $\sin r \neq 0$, $x_1(t)$ will grow unbounded. This internal stability issue will be addressed by using the concept of zero dynamics.

Relative Degree

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

where f, g, and h are sufficiently smooth in a domain D
$f : D \rightarrow R^n$ and $g : D \rightarrow R^n$ are called vector fields on D

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)\, u$$

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

is the *Lie Derivative* of h with respect to f or along f

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$$

$$L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x)$$

$$L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x)$$

$$L_f^0 h(x) = h(x)$$

$$\dot{y} = L_f h(x) + L_g h(x)\, u$$

$$L_g h(x) = 0 \;\; \Rightarrow \;\; \dot{y} = L_f h(x)$$

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x} [f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)\, u$$

$$L_g L_f h(x) = 0 \;\Rightarrow\; y^{(2)} = L_f^2 h(x)$$

$$y^{(3)} = L_f^3 h(x) + L_g L_f^2 h(x)\, u$$

$$L_g L_f^{i-1} h(x) = 0, \;\; i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0$$

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x)\, u$$

Definition: The system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ, $1 \leq \rho \leq n$, in $D_0 \subset D$ if $\forall\, x \in D_0$

$$L_g L_f^{i-1} h(x) = 0, \;\; i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0$$

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_1, \quad \varepsilon > 0$$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$$

Relative degree $= 2$ over R^2

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_2, \quad \varepsilon > 0$$

$$\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$$

Relative degree $= 1$ over R^2

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_1 + x_2^2, \quad \varepsilon > 0$$

$$\dot{y} = x_2 + 2x_2[-x_1 + \varepsilon(1 - x_1^2)x_2 + u]$$

Relative degree $= 1$ over $\{x_2 \neq 0\}$

Example: Field-controlled DC motor

$$\dot{x}_1 = -ax_1 + u, \; \dot{x}_2 = -bx_2 + k - cx_1x_3, \; \dot{x}_3 = \theta x_1 x_2, \; y = x_3$$

a, b, c, k, and θ are positive constants

$$\dot{y} = \dot{x}_3 = \theta x_1 x_2$$

$$\ddot{y} = \theta x_1 \dot{x}_2 + \theta \dot{x}_1 x_2 = (\cdot) + \theta x_2 u$$

Relative degree $= 2$ over $\{x_2 \neq 0\}$

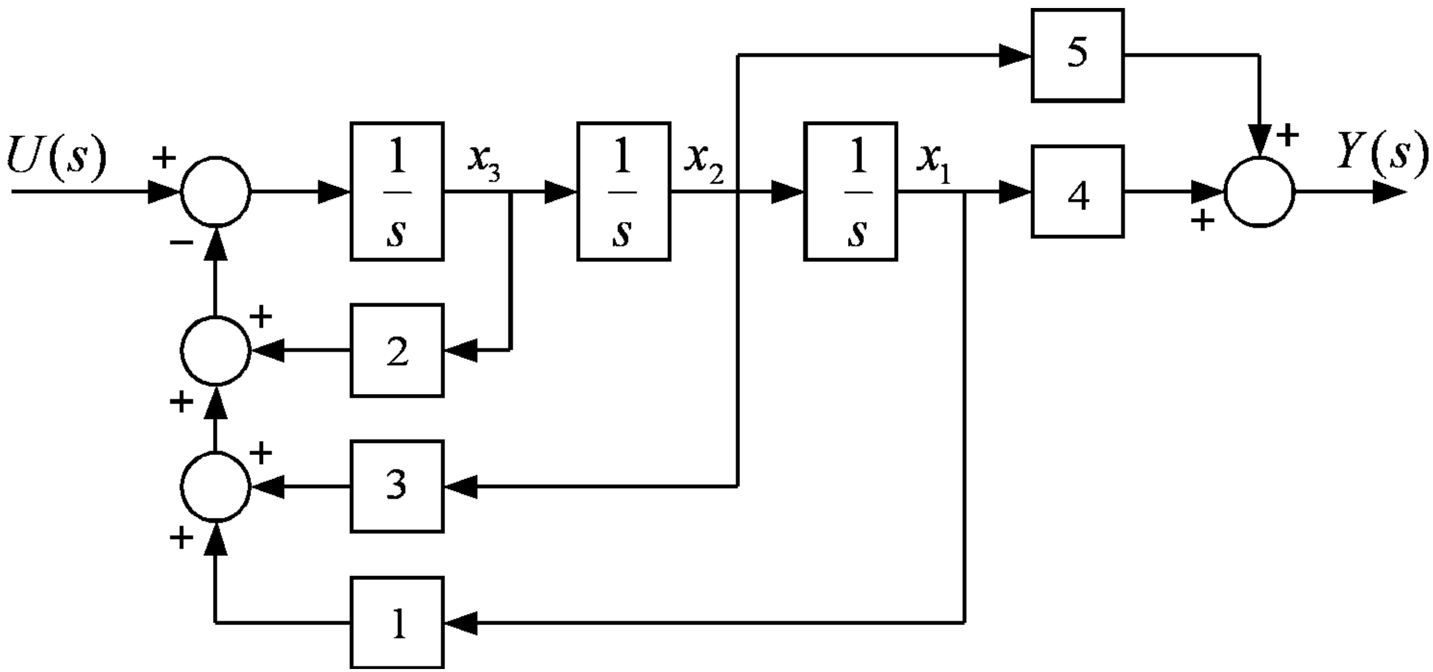
Relative Degree of LTI System

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5s+4}{s^3+2s^2+3s+1}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Relative Degree of LTI System



Normal Form

Change of variables:

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix}$$

ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on a domain $D_0 \subset D$

$$
\begin{aligned}
\dot{\eta} &= \frac{\partial \phi}{\partial x}[f(x) + g(x)u] = f_0(\eta, \xi) + g_0(\eta, \xi)u \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\, u \\
y &= \xi_1
\end{aligned}
$$

Choose $\phi(x)$ such that $T(x)$ is a diffeomorphism and

$$
\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \le i \le n - \rho, \ \forall\, x \in D_0
$$

Always possible (at least locally)

$$
\dot{\eta} = f_0(\eta, \xi)
$$

Theorem 13.1: Suppose the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ ($\leq n$) in D. If $\rho = n$, then for every $x_0 \in D$, a neighborhood N of x_0 exists such that the map $T(x) = \psi(x)$, restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in D$, a neighborhood N of x_0 and smooth functions $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ exist such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n - \rho$$

is satisfied for all $x \in N$ and the map $T(x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}$, restricted to N, is a diffeomorphism on N

Normal Form:

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\ \dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\, u \\ y &= \xi_1 \end{aligned}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\left[L_f^\rho h(x) + L_gL_f^{\rho-1}h(x)\ u\right] \\ y &= C_c\xi \end{aligned}$$

$$\gamma(x) = L_gL_f^{\rho-1}h(x), \quad \alpha(x) = -\frac{L_f^\rho h(x)}{L_gL_f^{\rho-1}h(x)}$$

$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)]$$

If x^* is an open-loop equilibrium point at which $y = 0$; i.e., $f(x^*) = 0$ and $h(x^*) = 0$, then $\psi(x*) = 0$. Take $\phi(x^*) = 0$ so that $z = 0$ is an open-loop equilibrium point.

Zero Dynamics

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)] \\ y &= C_c\xi\end{aligned}$$

$$y(t) \equiv 0 \;\Rightarrow\; \xi(t) \equiv 0 \;\Rightarrow\; u(t) \equiv \alpha(x(t)) \;\Rightarrow\; \dot{\eta} = f_0(\eta, 0)$$

Definition: The equation $\dot{\eta} = f_0(\eta, 0)$ is called the *zero dynamics* of the system. The system is said to be *minimum phase* if zero dynamics have an asymptotically stable equilibrium point in the domain of interest (at the origin if $T(0) = 0$)

The zero dynamics can be characterized in the x-coordinates

$$Z^* = \{x \in D_0 \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0\}$$

$$y(t) \equiv 0 \;\Rightarrow\; x(t) \in Z^*$$

$$\Rightarrow\; u = u^*(x) \stackrel{\text{def}}{=} \alpha(x)|_{x \in Z^*}$$

The restricted motion of the system is described by

$$\dot{x} = f^*(x) \stackrel{\text{def}}{=} [f(x) + g(x)\alpha(x)]_{x \in Z^*}$$

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_2$$

$$\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u \;\Rightarrow\; \rho = 1$$

$$y(t) \equiv 0 \;\Rightarrow x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_1 = 0$$

Non-minimum phase

Example

$$\dot{x}_1 = -x_1 + \frac{2 + x_3^2}{1 + x_3^2}\, u, \;\; \dot{x}_2 = x_3, \;\; \dot{x}_3 = x_1 x_3 + u, \;\; y = x_2$$

$$\dot{y} = \dot{x}_2 = x_3$$

$$\ddot{y} = \dot{x}_3 = x_1 x_3 + u \quad \Rightarrow \quad \rho = 2$$

$$\gamma = L_g L_f h(x) = 1, \;\; \alpha = -\frac{L_f^2 h(x)}{L_g L_f h(x)} = -x_1 x_3$$

$$Z^* = \{x_2 = x_3 = 0\}$$

$$u = u^*(x) = 0 \quad \Rightarrow \quad \dot{x}_1 = -x_1$$

Minimum phase

Find $\phi(x)$ such that

$$\phi(0) = 0, \;\; \frac{\partial \phi}{\partial x} g(x) = \begin{bmatrix} \frac{\partial \phi}{\partial x_1}, & \frac{\partial \phi}{\partial x_2}, & \frac{\partial \phi}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{2+x_3^2}{1+x_3^2} \\ 0 \\ 1 \end{bmatrix} = 0$$

and

$$T(x) = \begin{bmatrix} \phi(x) & x_2 & x_3 \end{bmatrix}^T$$

is a diffeomorphism

$$\frac{\partial \phi}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial \phi}{\partial x_3} = 0$$

$$\phi(x) = -x_1 + x_3 + \tan^{-1} x_3$$

$$T(x) = \begin{bmatrix} -x_1 + x_3 + \tan^{-1} x_3, & x_2, & x_3 \end{bmatrix}^T$$

is a global diffeomorphism

$$\eta = -x_1 + x_3 + \tan^{-1} x_3, \;\; \xi_1 = x_2, \;\; \xi_2 = x_3$$

$$\begin{aligned} \dot{\eta} &= (-\eta + \xi_2 + \tan^{-1} \xi_2) \left(1 + \frac{2 + \xi_2^2}{1 + \xi_2^2} \xi_2\right) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= (-\eta + \xi_2 + \tan^{-1} \xi_2)\, \xi_2 + u \\ y &= \xi_1 \end{aligned}$$