Chapter 13
Feedback Linearization

Input-State Linearization

Definition: A nonlinear system is in the controller form if

$$\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$$

where (A, B) is controllable and $\gamma(x)$ is a nonsingular

$$u = \alpha(x) + \gamma^{-1}(x)v \quad \Rightarrow \quad \dot{x} = Ax + Bv$$

The n-dimensional single-input (SI) system

$$\dot{x} = f(x) + g(x)u$$

can be transformed into the controller form if $\exists\ h(x)$ s.t.

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree n. Why?

Transform the system into the normal form

$$\dot{z} = A_c z + B_c \gamma(z)[u - \alpha(z)], \qquad y = C_c z$$

On the other hand, if there is a change of variables $\zeta = S(x)$ that transforms the SI system

$$\dot{x} = f(x) + g(x)u$$

into the controller form

$$\dot{\zeta} = A\zeta + B\gamma(\zeta)[u - \alpha(\zeta)]$$

then there is a function $h(x)$ such that the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree n. Why?

For any controllable pair (A, B), we can find a nonsingular matrix M that transforms (A, B) into a controllable canonical form:

$$MAM^{-1} = A_c + B_c\lambda^T, \qquad MB = B_c$$

$$z = M\zeta = MS(x) \stackrel{\text{def}}{=} T(x)$$

$$\dot{z} = A_c z + B_c \gamma(\cdot)[u - \alpha(\cdot)]$$

$$h(x) = T_1(x)$$

In summary, the n-dimensional SI system

$$\dot{x} = f(x) + g(x)u$$

is transformable into the controller form if and only if $\exists\ h(x)$ such that

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree n

Search for a smooth function $h(x)$ such that

$$L_g L_f^{i-1} h(x) = 0,\ \ i = 1, 2, \ldots, n-1, \ \text{ and } \ L_g L_f^{n-1} h(x) \neq 0$$

$$T(x) = \begin{bmatrix} h(x), & L_f h(x), & \cdots & L_f^{n-1} h(x) \end{bmatrix}$$

$$T_1(x) = h(x)$$

$$\dot{T}_1 = \nabla T_1(f(x) + g(x)u) = L_f T_1 + L_g T_1 \cdot u = L_f T_1 = T_2$$

$$\dot{T}_2 = \nabla T_2(f(x) + g(x)u) = \nabla(L_f T_1)(f(x) + g(x)u)$$

$$= L_f{}^2 T_1 + L_g L_f T_1 u = L_f{}^2 T_1 = T_3$$

$$\vdots$$

$$\dot{T}_{n-1} = L_f^{n-1} T_1 = T_n$$

$$\dot{T}_n = \nabla T_n(f(x) + g(x)u) = \nabla(L_f^{n-1} T_1)(f(x) + g(x)u)$$

$$= L_f{}^n T_1 + L_g L_f{}^{n-1} T_1 u = v$$

$$u = \frac{1}{L_g L_f{}^{n-1} T_1}\left(v - L_f{}^n T_1\right)$$

The Lie Bracket: For two vector fields f and g, the *Lie bracket* $[f,g]$ is a third vector field defined by

$$[f,g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

Notation:

$$ad_f^0 g(x) = g(x), \qquad ad_f g(x) = [f,g](x)$$

$$ad_f^k g(x) = [f, ad_f^{k-1} g](x), \;\; k \geq 1$$

Properties:

- $[f,g] = -[g,f]$
- For constant vector fields f and g, $[f,g] = 0$

Example

$$f = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

$$[f, g] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

$$ad_f g = [f, g] = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}$$

$$f = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \quad ad_f g = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}$$

$$ad_f^2 g = [f, ad_f g] =$$

$$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + x_2 - \sin x_1 - x_1 \cos x_1 \end{bmatrix}$$

Distribution: For vector fields f_1, f_2, ..., f_k on $D \subset R^n$, let

$$\Delta(x) = \mathrm{span}\{f_1(x), f_2(x), \ldots, f_k(x)\}$$

The collection of all vector spaces $\Delta(x)$ for $x \in D$ is called a *distribution* and referred to by

$$\Delta = \mathrm{span}\{f_1, f_2, \ldots, f_k\}$$

If $\dim(\Delta(x)) = k$ for all $x \in D$, we say that Δ is a nonsingular distribution on D, generated by f_1, ..., f_k

A distribution Δ is *involutive* if

$$g_1 \in \Delta \ \text{ and } \ g_2 \in \Delta \ \Rightarrow \ [g_1, g_2] \in \Delta$$

Lemma: If Δ is a nonsingular distribution, generated by $f_1, \ldots, f_k$, then it is involutive if and only if

$$[f_i, f_j] \in \Delta, \ \ \forall\, 1 \le i, j \le k$$

Example: $D = R^3$; $\Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \ \ f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \quad \dim(\Delta(x)) = 2, \ \forall\, x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\operatorname{rank}\,[f_1(x), f_2(x), [f_1, f_2](x)] =$$

$$\operatorname{rank}\begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \ \ \forall\, x \in D$$

Δ is not involutive

Example: $D = \{x \in R^3 \mid x_1^2 + x_3^2 \neq 0\}$; $\Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \; f_2 = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}, \; \dim(\Delta(x)) = 2, \; \forall\, x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} -4x_3 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{rank} \begin{bmatrix} 2x_3 & -x_1 & -4x_3 \\ -1 & -2x_2 & 2 \\ 0 & x_3 & 0 \end{bmatrix} = 2, \quad \forall\, x \in D$$

Δ is involutive

Theorem: The n-dimensional SI system

$$\dot{x} = f(x) + g(x)u$$

is transformable into the controller form if and only if there is a domain D_0 such that

$$\text{rank}[g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)] = n, \;\; \forall \, x \in D_0$$

and

$$\text{span} \; \{g, ad_f g, \ldots, ad_f^{n-2} g\} \text{ is involutive in } D_0$$

$$\dot{x} = Ax + Bu$$

$$f(x) = Ax, g(x) = B$$

$$ad_f g = \nabla g \cdot f - \nabla f \cdot g = -AB$$

$$ad_f{}^2 g = \nabla ad_f g \cdot f - \nabla f \cdot ad_f g = -A(-AB) = A^2 B$$

The Frobenius Theorem

Consider the set of first-order partial differential equations

$$\frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \frac{\partial h}{\partial x_3} f_3 = 0 \qquad \nabla h \cdot f = L_f h = 0 \qquad (6.44a)$$

$$\frac{\partial h}{\partial x_1} g_1 + \frac{\partial h}{\partial x_2} g_2 + \frac{\partial h}{\partial x_3} g_3 = 0 \qquad \nabla h \cdot g = L_g h = 0 \qquad (6.44b)$$

where $f_i(x_1, x_2, x_3)$ and $g_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ are known scalar functions of x_1, x_2, x_3, and $h(x_1, x_2, x_3)$ is an unknown function. Clearly, this set of partial differential equations is uniquely defined by the two vectors $\mathbf{f} = (f_1 \;\; f_2 \;\; f_3)^T$, $\mathbf{g} = (g_1 \;\; g_2 \;\; g_3)^T$. If a solution $h(x_1, x_2, x_3)$ exists for the above partial differential equations, we shall say the set of vector fields $\{\mathbf{f}, \mathbf{g}\}$ is *completely integrable*.

$$L_{f_j} h_i = 0$$

$$n = 3, m = 2, n - m = 1$$

$$i = 1, j = 1, 2$$

The Frobenius Theorem

The question now is to determine when these equations are solvable. This is not obvious at all, *a priori*. The Frobenius theorem provides a relatively simple condition: Equation (6.44) has a solution $h(x_1, x_2, x_3)$ if, and only if, there exists scalar functions $\alpha_1(x_1, x_2, x_3)$ and $\alpha_2(x_1, x_2, x_3)$ such that

$$[\mathbf{f}, \mathbf{g}] = \alpha_1 \mathbf{f} + \alpha_2 \mathbf{g}$$

i.e., if the Lie bracket of **f** and **g** can be expressed as a linear combination of **f** and **g**. This condition is called the *involutivity condition* on the vector fields $\{\mathbf{f}, \mathbf{g}\}$. Geometrically it means that the vector $[\mathbf{f}, \mathbf{g}]$ is in the plane formed by the two vectors **f** and **g**. Thus, the Frobenius theorem states that the set of vector fields $\{\mathbf{f}, \mathbf{g}\}$ is completely integrable if, and only if, it is involutive. Note that the involutivity condition can be relatively easily checked, and therefore, the solvability of (6.44) can be determined accordingly.

The Frobenius Theorem

Definition 6.4 *A linearly independent set of vector fields* $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ *on* $\mathbf{R}^n$ *is said to be* *<u>completely integrable</u>* *if, and only if, there exist* $n-m$ *scalar functions* $h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots, h_{n-m}(\mathbf{x})$ *satisfying the system of partial differential equations*

$$\nabla h_i \, \mathbf{f}_j = 0 \qquad\qquad L_{f_j} h_i = 0 \tag{6.45}$$

where $1 \le i \le n-m$, $1 \le j \le m$, *and the gradients* ∇h_i *are linearly independent.*

Definition 6.5 *A linearly independent set of vector fields* $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m\}$ *is said to be* *<u>involutive</u>* *if, and only if, there are scalar functions* $\alpha_{ijk} : \mathbf{R}^n \to \mathbf{R}$ *such that*

$$[\mathbf{f}_i, \mathbf{f}_j](\mathbf{x}) = \sum_{k=1}^{m} \alpha_{ijk}(\mathbf{x}) \, \mathbf{f}_k(\mathbf{x}) \qquad \forall\, i, j \tag{6.46}$$

Theorem 6.1 (Frobenius) *Let* $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_m$ *be a set of linearly independent vector fields. The set is completely integrable if, and only if, it is involutive.*

Example (Frobenius)

Example 6.7: Consider the set of partial differential equations

$$2x_3 \frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial x_2} = 0$$

$$-x_1 \frac{\partial h}{\partial x_1} - 2x_2 \frac{\partial h}{\partial x_2} + x_3 \frac{\partial h}{\partial x_3} = 0$$

The associated vector fields are $\{\mathbf{f}_1, \mathbf{f}_2\}$ with

$$\mathbf{f}_1 = (2x_3 \quad -1 \quad 0)^T \qquad \mathbf{f}_2 = (-x_1 \quad -2x_2 \quad x_3)^T$$

$$[\mathbf{f}_1, \mathbf{f}_2] = (-4x_3 \quad 2 \quad 0)^T$$

$$[\mathbf{f}_1, \mathbf{f}_2] = -2\,\mathbf{f}_1 + 0\,\mathbf{f}_2$$

Example

$$\dot{x} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} -a \cos x_2 \\ 0 \end{bmatrix}$$

$$[g(x), ad_f g(x)] = \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix}$$

$\mathrm{rank}[g(x), ad_f g(x)] = 2, \ \forall\, x$ such that $\cos x_2 \neq 0$

$\mathrm{span}\{g\}$ is involutive

Find h such that $L_g h(x) = 0,$ and $L_g L_f h(x) \neq 0$

$$\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0 \quad \Rightarrow \quad h \text{ is independent of } x_2$$

$$L_f h(x) = \frac{\partial h}{\partial x_1} a \sin x_2$$

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos x_2$$

$L_g L_f h(x) \neq 0$ in $D_0 = \{x \in R^2 | \cos x_2 \neq 0\}$ if $\frac{\partial h}{\partial x_1} \neq 0$

Take $h(x) = x_1 \quad \Rightarrow \quad T(x) = \begin{bmatrix} h \\ L_f h \end{bmatrix} = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix}$

How to Perform Input-State Linearization

- Construct the vector fields $\mathbf{g}$, $ad_{\mathbf{f}}\,\mathbf{g}, \ldots, ad_{\mathbf{f}}^{\,n-1}\mathbf{g}$ for the given system
- Check whether the controllability and involutivity conditions are satisfied
- If both are satisfied, find the first state T_1 (the output function leading to input-output linearization of relative degree n) from equations (6.53), *i.e.*,

$$\nabla T_1 \, ad_{\mathbf{f}}^{\,i}\mathbf{g} = 0 \quad i = 0, \ldots, n-2 \tag{6.57a}$$

$$\nabla T_1 \, ad_{\mathbf{f}}^{\,n-1}\mathbf{g} \neq 0 \tag{6.57b}$$

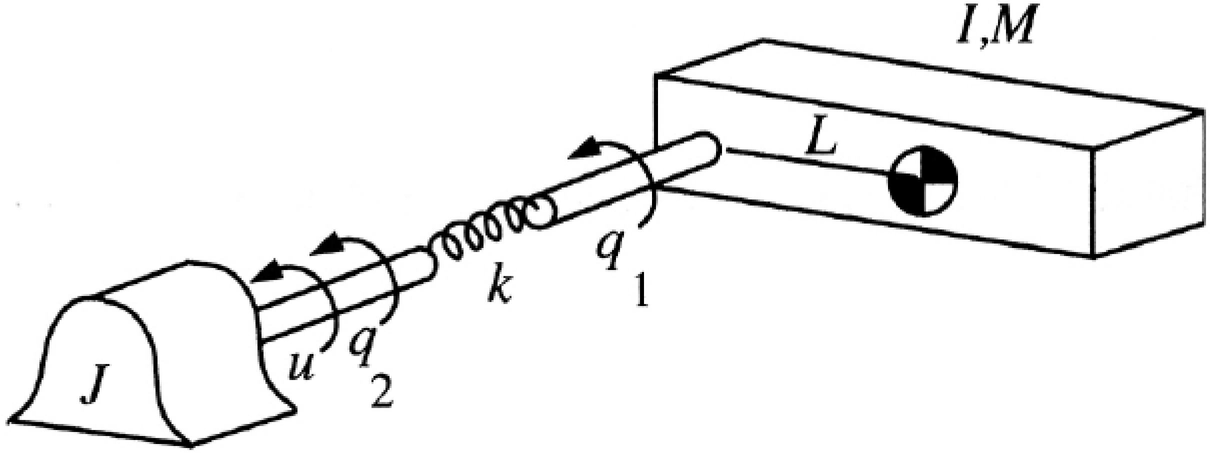
How to Perform Input-State Linearization

• Compute the state transformation $\mathbf{z} = \boldsymbol{\phi}(\mathbf{x}) = (\,T_1 \quad L_{\mathbf{f}}\,T_1 \quad \ldots \quad L_{\mathbf{f}}{}^{n-1}T_1\,)^T$ and the input transformation (6.48), with

$$\alpha(\mathbf{x}) = -\frac{L_{\mathbf{f}}{}^{n}T_1}{L_{\mathbf{g}}\,L_{\mathbf{f}}{}^{n-1}T_1} \tag{6.58a}$$

$$\beta(\mathbf{x}) = \frac{1}{L_{\mathbf{g}}\,L_{\mathbf{f}}{}^{n-1}T_1} \tag{6.58b}$$

Example 6.8



$$I\ddot{q}_1 + MgL\sin q_1 + k(q_1 - q_2) = 0$$

$$J\ddot{q}_2 - k(q_1 - q_2) = u$$

Example 6.8

$$\mathbf{x} = [q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2]^T$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u$$

Example 6.8

$$\mathbf{f} = [x_2 \quad -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \quad x_4 \quad \frac{k}{J}(x_1 - x_3)]^T$$

$$\mathbf{g} = [0 \quad 0 \quad 0 \quad \frac{1}{J}]^T$$

$$[\mathbf{g}, ad_{\mathbf{f}}\mathbf{g}, ad^2{}_{\mathbf{f}}\mathbf{g}, ad^3{}_{\mathbf{f}}\mathbf{g}] = \begin{bmatrix} 0 & 0 & 0 & -\frac{k}{IJ} \\ 0 & 0 & \frac{k}{IJ} & 0 \\ 0 & -\frac{1}{J} & 0 & \frac{k}{J^2} \\ \frac{1}{J} & 0 & -\frac{k}{J^2} & 0 \end{bmatrix}$$

It has rank 4 for $k > 0, IJ < \infty$. Furthermore, since the vector fields $\{\mathbf{g}, ad_{\mathbf{f}}\,\mathbf{g}, ad_{\mathbf{f}}{}^2\mathbf{g}\}$ are constant, they form an involutive set. Therefore, the system in (6.59) is input-state linearizable.

Example 6.8

$$\nabla T_1 = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{bmatrix}$$

$$\nabla T_1 ad_f^{\,0} g = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix} = 0$$

$$\rightarrow \frac{\partial T_1}{\partial x_4} = 0$$

Example 6.8

$$\nabla T_1 = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{bmatrix}$$

$$\nabla T_1 ad_f^{\,3} g = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \frac{\partial T_1}{\partial x_3} & \frac{\partial T_1}{\partial x_4} \end{bmatrix} \begin{bmatrix} -\frac{k}{IJ} \\ 0 \\ \frac{k}{J^2} \\ 0 \end{bmatrix} \neq 0$$

$$\rightarrow \frac{\partial T_1}{\partial x_1} \neq 0$$

Example 6.8

$$\frac{\partial T_1}{\partial x_2} = 0 \qquad \frac{\partial T_1}{\partial x_3} = 0 \qquad \frac{\partial T_1}{\partial x_4} = 0$$

$$\frac{\partial T_1}{\partial x_1} \neq 0 \qquad z_1 = T_1 = x_1$$

$$z_2 = T_2 = L_f T_1 = \nabla T_1 f$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} = x_2$$

Example 6.8

$$z_3 = T_3 = L_f{}^2 T_1 = L_f(L_f T_1) = \nabla T_2 f$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix} = -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)$$

Example 6.8

$$z_4 = T_4 = L_f{}^3 T_1 = L_f (L_f{}^2 T_1) = \nabla T_3 f$$

$$= \begin{bmatrix} -\frac{MgL}{I}\cos x_1 - \frac{k}{I} & 0 & \frac{k}{I} & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{MgL}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\ x_4 \\ \frac{k}{J}(x_1 - x_3) \end{bmatrix}$$

$$= -\frac{MgL}{I} x_2 \cos x_1 - \frac{k}{I}(x_2 - x_4)$$

Example 6.8

$$z_2 = T_2 = \frac{\partial T_1}{\partial \mathbf{x}} \mathbf{f} = x_2$$

$$z_3 = T_3 = \frac{\partial T_2}{\partial \mathbf{x}} \mathbf{f} = -\frac{MgL}{I} \sin x_1 - \frac{k}{I} (x_1 - x_3)$$

$$z_4 = T_4 = \frac{\partial T_3}{\partial \mathbf{x}} \mathbf{f} = -\frac{MgL}{I} x_2 \cos x_1 - \frac{k}{I} (x_2 - x_4)$$

Example 6.8

$$u = \frac{\nu - L_f{}^4 T_1}{L_g L_f{}^3 T_1}$$

$$L_g L_f{}^3 T_1 = L_g T_4 = \nabla T_4 g$$

$$= \begin{bmatrix} \frac{\partial T_4}{\partial x_1} & \frac{\partial T_4}{\partial x_2} & \frac{\partial T_4}{\partial x_3} & \frac{\partial T_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix} = \frac{k}{IJ}$$

Example 6.8

$$L_f^{\,4} T_1 = L_f (L_f^{\,3} T_1) = \nabla T_4 f$$

$$= \begin{bmatrix} \dfrac{\partial T_4}{\partial x_1} & \dfrac{\partial T_4}{\partial x_2} & \dfrac{\partial T_4}{\partial x_3} & \dfrac{\partial T_4}{\partial x_4} \end{bmatrix} \begin{bmatrix} x_2 \\ -\dfrac{MgL}{I} \sin x_1 - \dfrac{k}{I}(x_1 - x_3) \\ x_4 \\ \dfrac{k}{J}(x_1 - x_3) \end{bmatrix}$$

Example 6.8

$$u = \frac{IJ}{k}\,(\,v - a(\mathbf{x})\,)$$

$$a(\mathbf{x}) = \frac{MgL}{I}\sin x_1\,(x_2{}^2 + \frac{MgL}{I}\cos x_1 + \frac{k}{I}\,) \;+\; \frac{k}{I}(x_1 - x_3)\,(\,\frac{k}{I} + \frac{k}{J} + \frac{MgL}{I}\;\cos x_1\,)$$

Controller Design Based on Input-State Linearization

$$z_1^{(4)} = v$$

$$v = z_{d1}^{(4)} - \beta_3 \tilde{z}_1^{(3)} - \beta_2 \tilde{z}_1^{(2)} - \beta_1 \dot{\tilde{z}}_1 - \beta_o \tilde{z}_1$$

$$\tilde{z}_1^{(4)} + \beta_3 \tilde{z}_1^{(3)} + \beta_2 \tilde{z}_1^{(2)} + \beta_1 \dot{\tilde{z}}_1 + \beta_o \tilde{z}_1 = 0$$