Chapter 14
Nonlinear Design Tools

Sliding Mode Control

Backstepping

Sliding Mode Control

Example

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) + g(x)u, \quad g(x) \geq g_0 > 0$$

Sliding Manifold (Surface):

$$s = a_1 x_1 + x_2 = 0$$

$$s(t) \equiv 0 \;\; \Rightarrow \;\; \dot{x}_1 = -a_1 x_1$$

$$a_1 > 0 \;\; \Rightarrow \;\; \lim_{t \to \infty} x_1(t) = 0$$

How can we bring the trajectory to the manifold $s = 0$?

How can we maintain it there?

$$\dot{s} = a_1\dot{x}_1 + \dot{x}_2 = a_1 x_2 + h(x) + g(x)u$$

Suppose

$$\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq \varrho(x)$$

$$V = \tfrac{1}{2}s^2$$

$$\dot{V} = s\dot{s} = s[a_1 x_2 + h(x)] + g(x)su \leq g(x)|s|\varrho(x) + g(x)su$$

$$\beta(x) \geq \varrho(x) + \beta_0, \quad \beta_0 > 0$$

$$s > 0, \quad u = -\beta(x)$$

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$s < 0, \quad u = \beta(x)$$

$$\dot{V} \leq g(x)|s|\varrho(x) + g(x)su = g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$\text{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

$$u = -\beta(x)\,\text{sgn}(s)$$

$$\dot{V} \leq -g(x)\beta_0|s| \leq -g_0\beta_0|s|$$

$$\dot{V} \leq -g_0\beta_0\sqrt{2V}$$

$$\dot{V} \leq -g_0\beta_0\sqrt{2V}$$

$$\frac{dV}{\sqrt{V}} \leq -g_0\beta_0\sqrt{2}\,dt$$

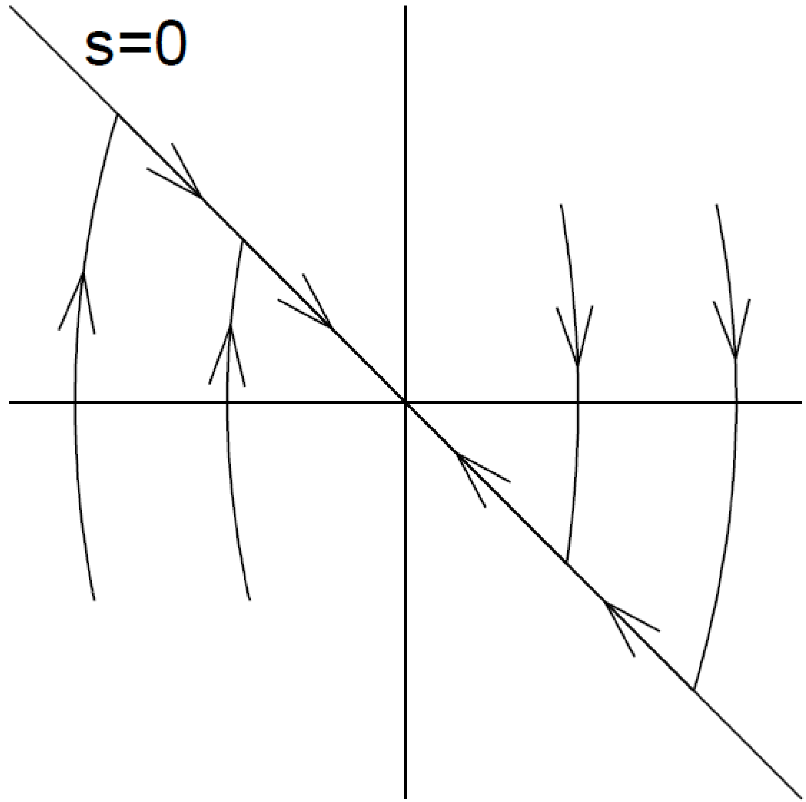
$$2\sqrt{V}\,\Big|_{V(s(0))}^{V(s(t))} \leq -g_0\beta_0\sqrt{2}\,t$$

$$\sqrt{V(s(t))} \leq \sqrt{V(s(0))} - g_0\beta_0\frac{1}{\sqrt{2}}\,t$$

$$|s(t)| \leq |s(0)| - g_0\beta_0\,t$$

$s(t)$ reaches zero in finite time

Once on the surface $s = 0$, the trajectory cannot leave it



What is the region of validity?

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) - g(x)\beta(x)\mathrm{sgn}(s)$$

$$\dot{x}_1 = -a_1 x_1 + s \qquad \dot{s} = a_1 x_2 + h(x) - g(x)\beta(x)\mathrm{sgn}(s)$$

$$s\dot{s} \le -g_0\beta_0|s|, \quad \text{if } \beta(x) \ge \varrho(x) + \beta_0$$

$$V_1 = \tfrac{1}{2}x_1^2$$

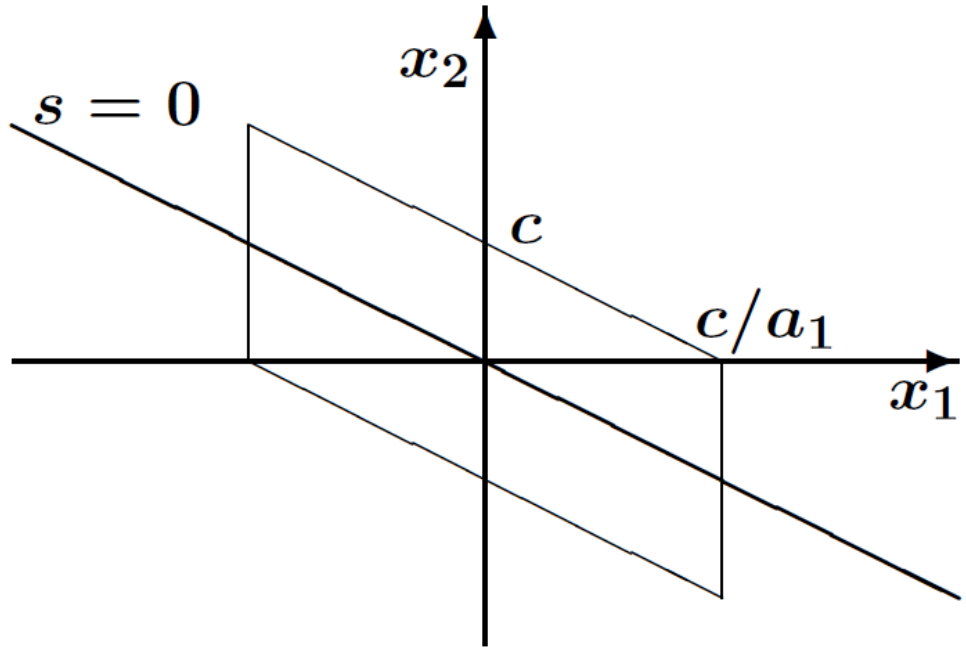
$$\dot{V}_1 = x_1\dot{x}_1 = -a_1x_1^2 + x_1 s \le -a_1x_1^2 + |x_1|c \le 0$$

$$\forall\ |s| \le c \text{ and } |x_1| \ge \frac{c}{a_1}$$

$$\Omega = \left\{ |x_1| \le \frac{c}{a_1},\ |s| \le c \right\}$$

Ω is positively invariant if $\left|\dfrac{a_1x_2 + h(x)}{g(x)}\right| \le \varrho(x)$ over Ω

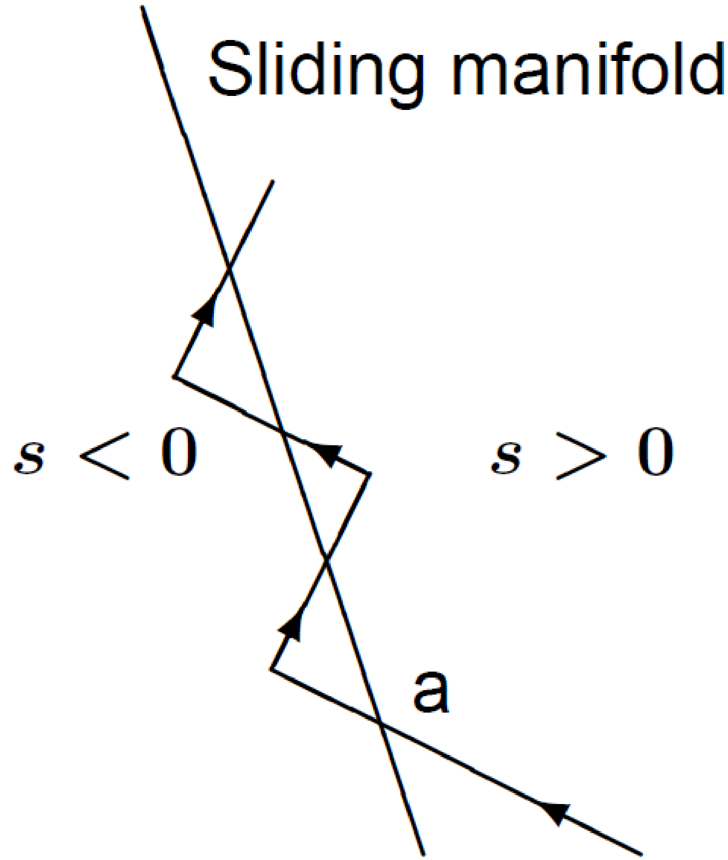
$$\Omega = \left\{ |x_1| \le \frac{c}{a_1},\ |s| \le c \right\}$$



$$\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \le k_1 < k, \ \ \forall \, x \in \Omega$$

$$u = -k \, \mathrm{sgn}(s)$$

Chattering



How can we reduce or eliminate chattering?

pendulum equation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(g_0/\ell)\sin(x_1+\delta_1) - (k_0/m)x_2 + (1/m\ell^2)u \\ u &= -k\ \text{sgn}(s) \;=\; -k\ \text{sgn}(a_1x_1+x_2)\end{aligned}$$

to stabilize the pendulum at $\delta_1 = \pi/2$, where $x_1 = \theta - \delta_1$ and $x_2 = \dot{\theta}$. The constants m, ℓ, k_0, and g_0 are the mass, length, coefficient of friction, and acceleration due to gravity, respectively. We take $a_1 = 1$ and $k = 4$. The gain $k = 4$ is chosen by using

$$\begin{aligned}\left|\frac{a_1x_2 + h(x)}{g}\right| &= \left|\ell^2(m-k_0)x_2 - mg_0\ell\cos(x_1)\right| \\ &\le \ell^2|m-k_0|(2\pi) + mg_0\ell \;\le\; 3.68\end{aligned}$$

where the bound is calculated over the set $\{|x_1| \le \pi,\ |x_1+x_2| \le \pi\}$ for $0.05 \le m \le 0.2$, $0.9 \le \ell \le 1.1$, and $0 \le k_0 \le 0.05$. The simulation is performed by using $m = 0.1$, $\ell = 1$, and $k_0 = 0.02$. Figure 14.4 shows ideal sliding mode control, while Figure 14.5 shows a nonideal case where switching is delayed by unmodeled actuator dynamics having the transfer function $1/(0.01s+1)^2$.

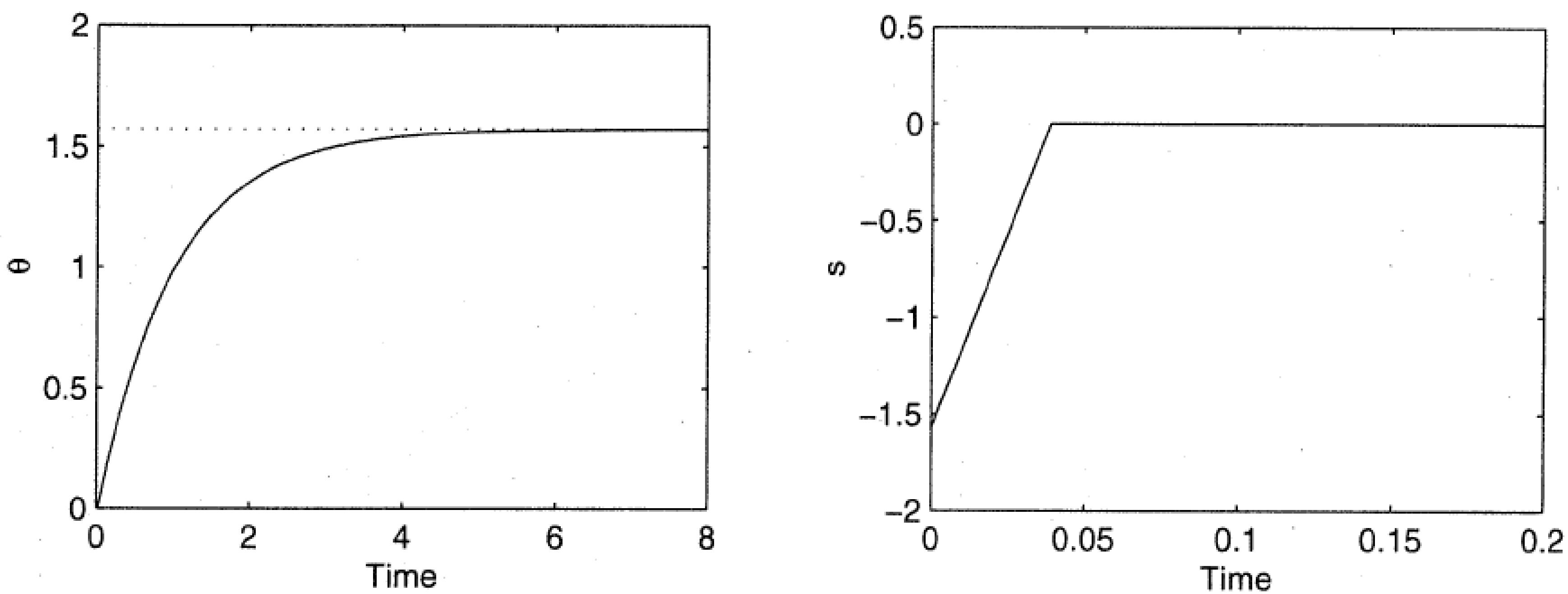


Figure 14.4: Ideal sliding mode control.

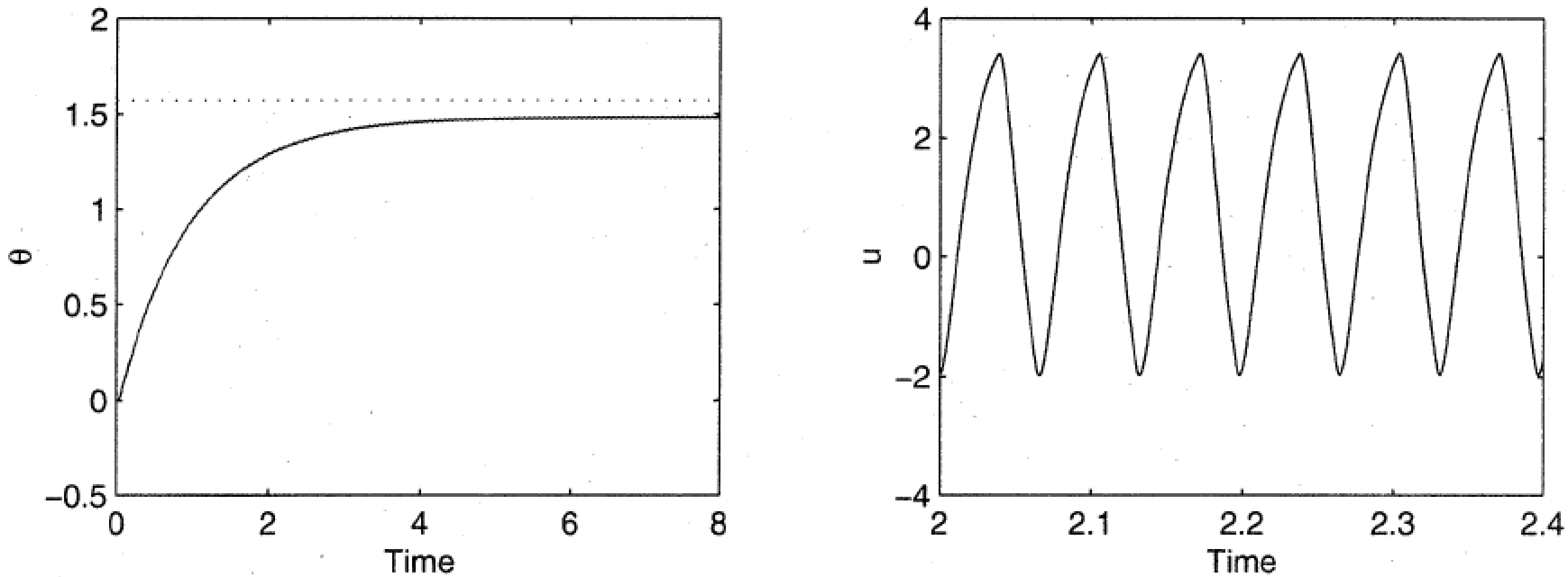


Figure 14.5: Sliding mode control with unmodeled actuator dynamics.

Reduce the amplitude of the signum function

$$\dot{s} = a_1 x_2 + h(x) + g(x)u$$

$$u = -\frac{[a_1 x_2 + \hat{h}(x)]}{\hat{g}(x)} + v$$

$$\dot{s} = \delta(x) + g(x)v$$

$$\delta(x) = a_1 \left[1 - \frac{g(x)}{\hat{g}(x)}\right] x_2 + h(x) - \frac{g(x)}{\hat{g}(x)}\hat{h}(x)$$

$$\left|\frac{\delta(x)}{g(x)}\right| \le \varrho(x), \qquad \beta(x) \ge \varrho(x) + \beta_0$$

$$v = -\beta(x)\,\mathrm{sgn}(s)$$

amplitude of the switching component would be smaller. For example, returning to the pendulum equation and taking $\hat{m} = 0.125$, $\hat{\ell} = 1$, $\hat{k}_0 = 0.025$ to be nominal values of m, ℓ, k_0, we have

$$\left|\frac{\delta(x)}{g}\right| = \left|\left(a_1 m\ell^2 - a_1 \hat{m}\hat{\ell}^2 - k_0\ell^2 + \hat{k}_0\hat{\ell}^2\right) x_2 - g_0(m\ell - \hat{m}\hat{\ell})\cos x_1\right| \leq 1.83$$

where the bound is calculated over the same set as before. The modified sliding mode control is taken as

$$u = -0.1x_2 + 1.2263\cos x_1 - 2\ \mathrm{sgn}(s)$$

which shows a reduction in the switching term amplitude from 4 to 2. Figure 14.6 shows simulation of this modified control in the presence of unmodeled actuator dynamics. The reduction in the amplitude of chattering is clear.

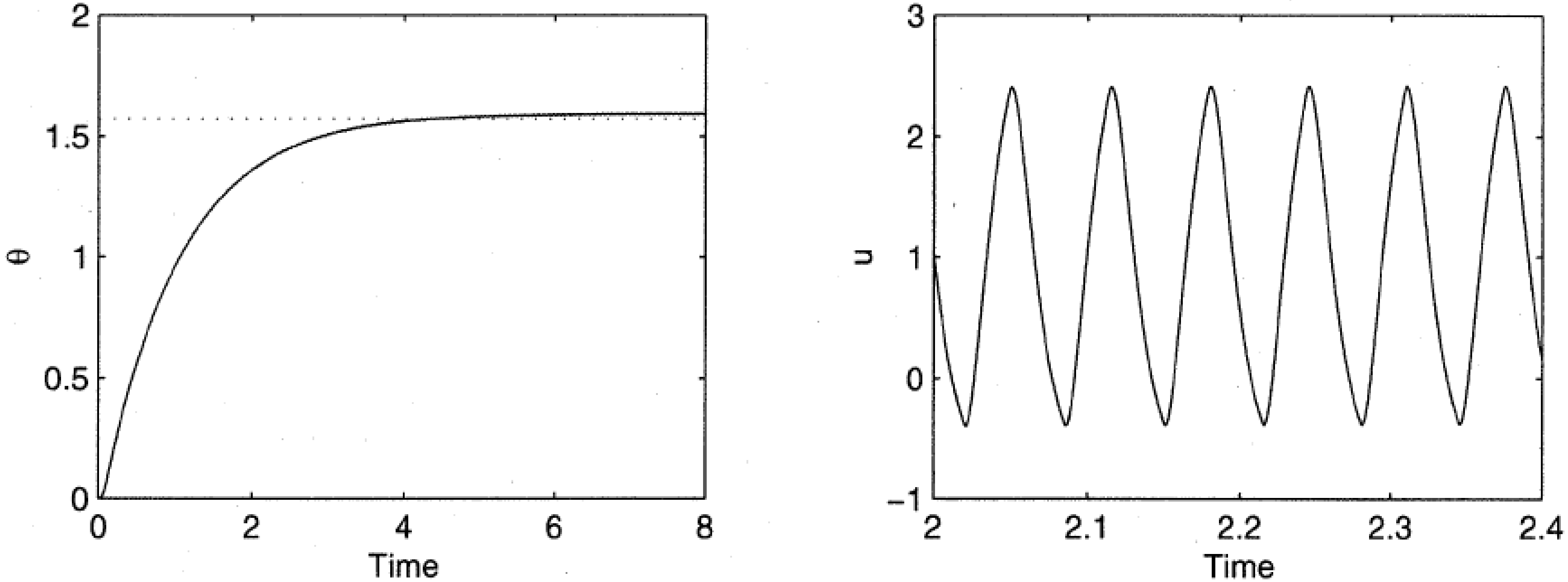
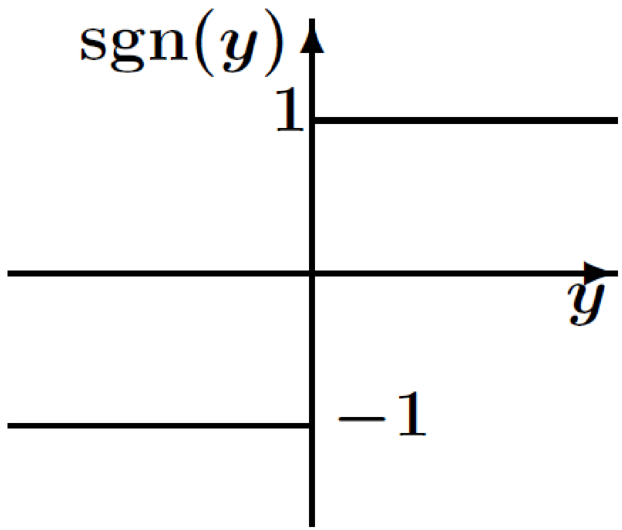


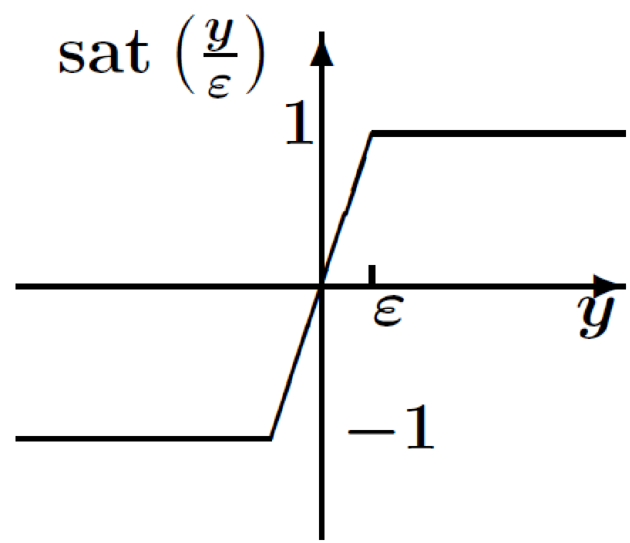
Figure 14.6: Modified sliding mode control with unmodeled actuator dynamics.

Replace the signum function by a high-slope saturation function

$$u = -\beta(x)\ \mathrm{sat}\left(\frac{s}{\varepsilon}\right)$$

$$\mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \mathrm{sgn}(y), & \text{if } |y| > 1 \end{cases}$$





How can we analyze the system?

$$\text{For } |s| \geq \varepsilon, \quad u = -\beta(x)\,\mathrm{sgn}(s)$$

With $c \geq \varepsilon$

- $\Omega = \left\{ |x_1| \leq \frac{c}{a_1},\ |s| \leq c \right\}$ is positively invariant
- The trajectory reaches the boundary layer $\{|s| \leq \varepsilon\}$in finite time
- The boundary layer is positively invariant

Inside the boundary layer:

$$\dot{x}_1 = -a_1 x_1 + s \qquad \dot{s} = a_1 x_2 + h(x) - g(x)\beta(x)\frac{s}{\varepsilon}$$

$$x_1 \dot{x}_1 \le -a_1 x_1^2 + |x_1|\varepsilon$$

$$0 < \theta < 1$$

$$x_1 \dot{x}_1 \le -(1-\theta)a_1 x_1^2, \;\; \forall\, |x_1| \ge \frac{\varepsilon}{\theta a_1}$$

The trajectories reach the positively invariant set

$$\Omega_\varepsilon = \{|x_1| \le \frac{\varepsilon}{\theta a_1}, \; |s| \le \varepsilon\}$$

in finite time

$\varepsilon\}$ in finite time. In general, we do not stabilize the origin, but we achieve ultimate boundedness with an ultimate bound that can be reduced by decreasing ε. What happens inside Ω_ε is problem dependent. Let us consider again the pendulum equation and see what happens inside Ω_ε in that case. Inside the boundary layer $\{|s| \leq \varepsilon\}$, the control reduces to the linear feedback law $u = -ks/\varepsilon$, and the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(g_0/\ell)\sin(x_1 + \delta_1) - (k_0/m)x_2 - (k/m\ell^2\varepsilon)(a_1 x_1 + x_2) \end{aligned}$$

has a unique equilibrium point at $(\bar{x}_1, 0)$, where $\bar{x}_1$ satisfies the equation

$$\varepsilon m g_0 \ell \sin(\bar{x}_1 + \delta_1) + k a_1 \bar{x}_1 = 0$$

and can be approximated for small ε by $\bar{x}_1 \approx -(\varepsilon m g_0 \ell / k a_1) \sin \delta_1$. Shifting the

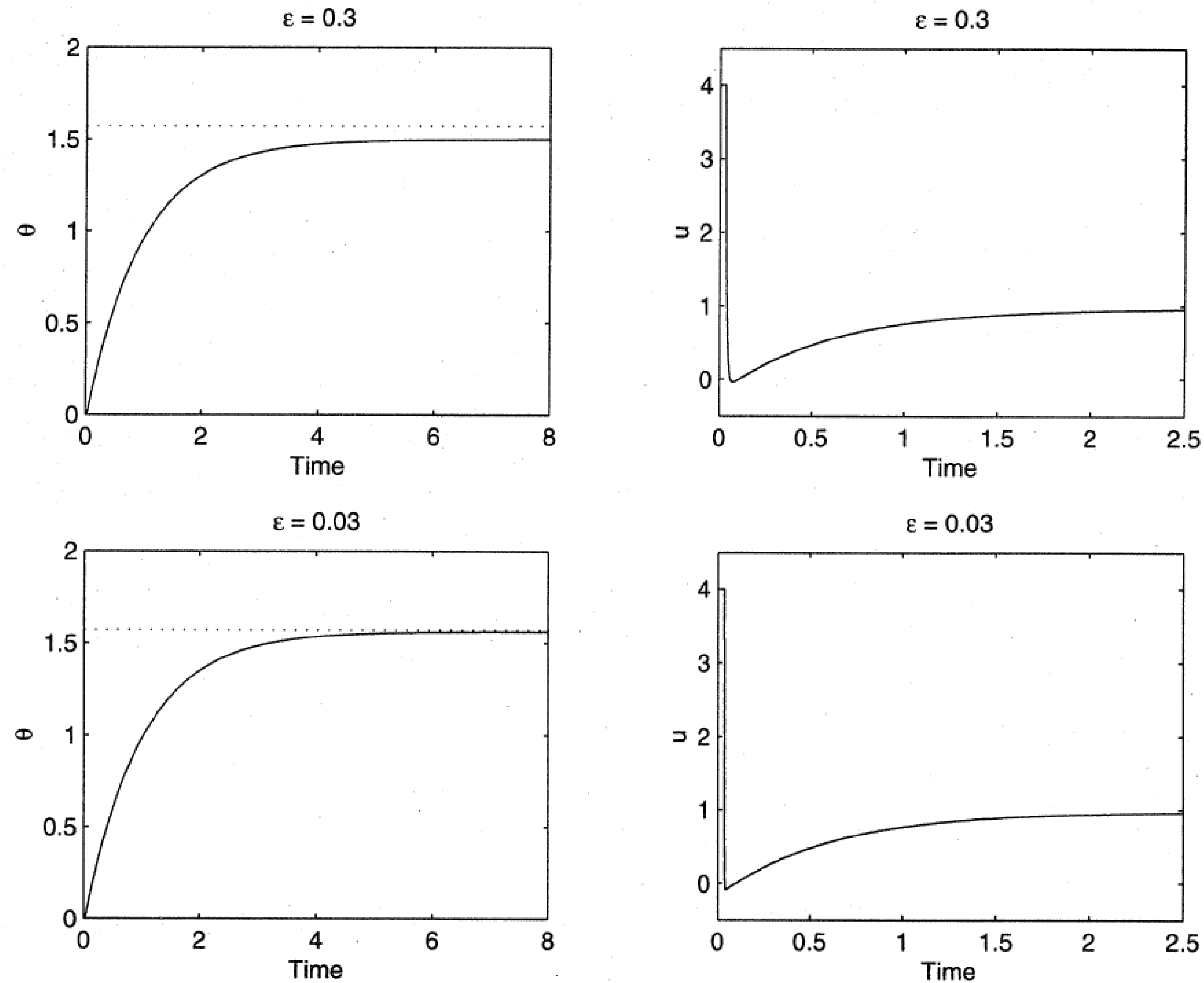


Figure 14.8: "Continuous" sliding mode control.

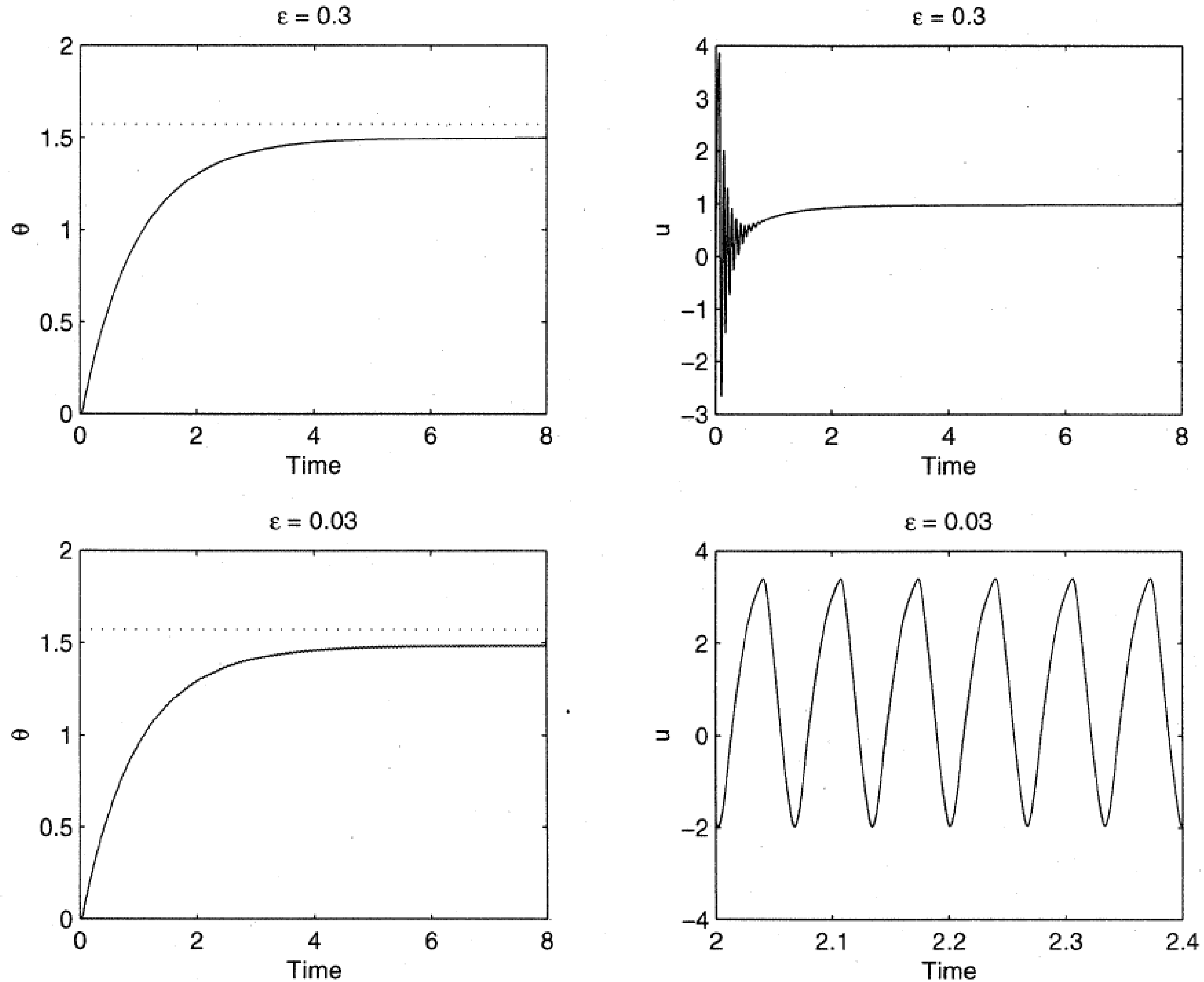


Figure 14.9: "Continuous" sliding mode control with unmodeled actuator dynamics.

Stabilization by Sliding Mode Control

Regular Form:

$$\begin{aligned}\dot{\eta} &= f_a(\eta, \xi) \\ \dot{\xi} &= f_b(\eta, \xi) + g(\eta, \xi)u + \delta(t, \eta, \xi, u)\end{aligned}$$

$$\eta \in R^{n-1},\ \xi \in R,\ u \in R$$

$$f_a(0,0) = 0,\ f_b(0,0) = 0,\ g(\eta, \xi) \geq g_0 > 0$$

Sliding Manifold:

$$s = \xi - \phi(\eta) = 0, \qquad \phi(0) = 0$$

$$s(t) \equiv 0 \ \Rightarrow \ \dot{\eta} = f_a(\eta, \phi(\eta))$$

Design ϕ s.t. the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is asymp. stable

$$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + g(\eta, \xi) u + \delta(t, \eta, \xi, u)$$

$$u = -\frac{1}{\hat{g}} \left(\hat{f}_b - \frac{\partial \phi}{\partial \eta} \hat{f}_a \right) + v \quad \textit{or} \quad u = v$$

$$u = -L \left(\hat{f}_b - \frac{\partial \phi}{\partial \eta} \hat{f}_a \right) + v, \quad L = \frac{1}{\hat{g}} \quad \textit{or} \quad L = 0$$

$$\dot{s} = g(\eta, \xi) v + \Delta(t, \eta, \xi, v)$$

$$\Delta = f_b - \frac{\partial \phi}{\partial \eta} f_a + \delta - gL \left(\hat{f}_b - \frac{\partial \phi}{\partial \eta} \hat{f}_a \right)$$

$$\left| \frac{\Delta(t, \eta, \xi, v)}{g(\eta, \xi)} \right| \le \varrho(\eta, \xi) + \kappa_0 |v|$$

$$\left|\frac{\Delta(t,\eta,\xi,v)}{g(\eta,\xi)}\right| \le \varrho(\eta,\xi) + \kappa_0|v|$$

$$\varrho(\eta,\xi) \ge 0, \quad 0 \le \kappa_0 < 1 \quad (Known)$$

$$s\dot{s} = sgv + s\Delta \le sgv + |s|\ |\Delta|$$

$$s\dot{s} \le g[sv + |s|(\varrho + \kappa_0|v|)]$$

$$v = -\beta(\eta,\xi)\ \text{sgn}(s)$$

$$\beta(\eta,\xi) \ge \frac{\varrho(\eta,\xi)}{1-\kappa_0} + \beta_0, \quad \beta_0 > 0$$

$$s\dot{s} \le g[-\beta|s| + \varrho|s| + \kappa_0\beta|s|] = g[-\beta(1-\kappa_0)|s| + \varrho|s|\ ]$$

$$s\dot{s} \le g[-\varrho|s| - (1-\kappa_0)\beta_0|s| + \varrho|s|\ ]$$

$$s\dot{s} \le -g(\eta,\xi)(1-\kappa_0)\beta_0|s| \le -g_0\beta_0(1-\kappa_0)|s|$$

$$v = -\beta(x)\,\mathrm{sat}\left(\frac{s}{\varepsilon}\right), \qquad \varepsilon > 0$$

$$s\dot{s} \le -g_0\beta_0(1-\kappa_0)|s|, \quad \text{for } |s| \ge \varepsilon$$

The trajectory reaches the boundary layer $\{|s| \le \varepsilon\}$ in finite time and remains inside thereafter

Study the behavior of η

$$\dot{\eta} = f_a(\eta, \phi(\eta) + s)$$

What do we know about this system and what do we need?

$$\alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|)$$

$$\frac{\partial V}{\partial \eta} f_a(\eta, \phi(\eta) + s) \leq -\alpha_3(\|\eta\|), \quad \forall \; \|\eta\| \geq \gamma(|s|)$$

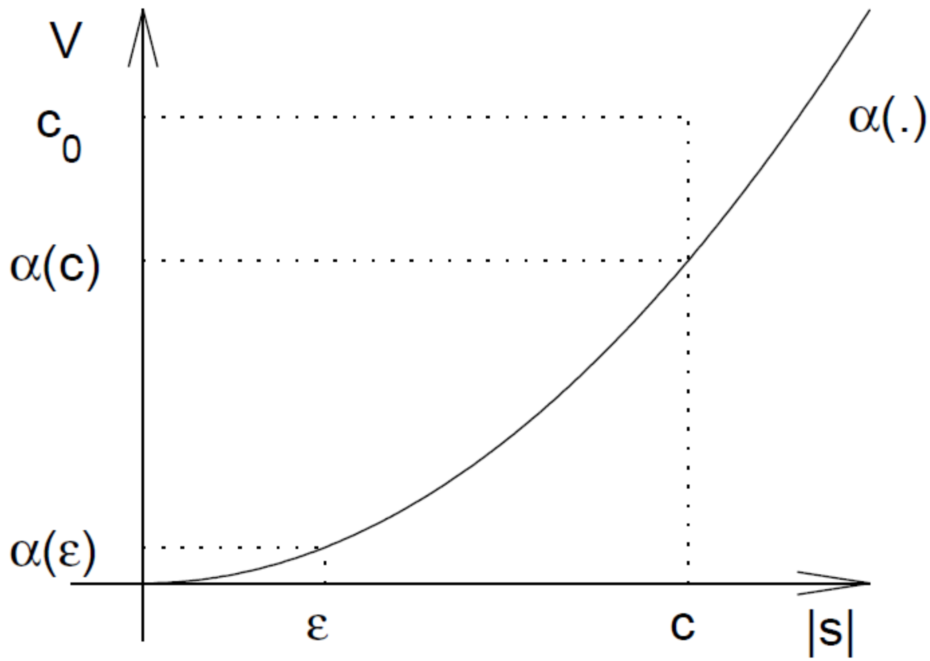
$$|s| \leq c \;\Rightarrow\; \dot{V} \leq -\alpha_3(\|\eta\|), \;\text{ for } \|\eta\| \geq \gamma(c)$$

$$\alpha(r) = \alpha_2(\gamma(r))$$

$$V(\eta) \geq \alpha(c) \Leftrightarrow V(\eta) \geq \alpha_2(\gamma(c)) \;\Rightarrow \alpha_2(\|\eta\|) \geq \alpha_2(\gamma(c))$$
$$\Rightarrow \|\eta\| \geq \gamma(c) \;\Rightarrow\; \dot{V} \leq -\alpha_3(\|\eta\|) \leq -\alpha_3(\gamma(c))$$

The set $\{V(\eta) \leq c_0\}$ with $c_0 \geq \alpha(c)$ is positively invariant

$$\Omega = \{V(\eta) \leq c_0\} \times \{|s| \leq c\}, \text{ with } c_0 \geq \alpha(c)$$



$$\Omega = \{V(\eta) \leq c_0\} \times \{|s| \leq c\}, \text{ with } c_0 \geq \alpha(c)$$

is positively invariant and all trajectories starting in Ω reach $\Omega_\varepsilon = \{V(\eta) \leq \alpha(\varepsilon)\} \times \{|s| \leq \varepsilon\}$ in finite time

Theorem 14.1: Suppose all the assumptions hold over Ω. Then, for all $(\eta(0), \xi(0)) \in \Omega$, the trajectory $(\eta(t), \xi(t))$ is bounded for all $t \geq 0$ and reaches the positively invariant set Ω_ε in finite time. If the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state

Theorem 14.2: Suppose all the assumptions hold over Ω

- $\varrho(0) = 0$, $\kappa_0 = 0$
- The origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stale

Then there exits $\varepsilon^* > 0$ such that for all $0 < \varepsilon < \varepsilon^*$, the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction. If the assumptions hold globally, the origin will be globally uniformly asymptotically stable

Example

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

$$|\theta_1| \le a, \quad |\theta_2| \le b$$

$$x_2 = -k x_1 \;\; \Rightarrow \;\; \dot{x}_1 = -k x_1 + \theta_1 x_1 \sin x_2$$

$$V_1 = \tfrac{1}{2} x_1^2 \;\; \Rightarrow \;\; x_1 \dot{x}_1 \le -k x_1^2 + a x_1^2$$

$$s = x_2 + k x_1, \quad k > a$$

$$\dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2)$$

$$u = -x_1 - k x_2 + v \;\; \Rightarrow \;\; \dot{s} = v + \Delta(x)$$

$$\Delta(x) = \theta_2 x_2^2 + k \theta_1 x_1 \sin x_2$$

$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$

$$|\Delta(x)| \le ak|x_1| + bx_2^2$$

$$\beta(x) = ak|x_1| + bx_2^2 + \beta_0, \quad \beta_0 > 0$$

$$u = -x_1 - kx_2 - \beta(x)\,\mathrm{sgn}(s)$$

Backstepping

$$\begin{aligned}\dot{\eta} &= f(\eta) + g(\eta)\xi \\ \dot{\xi} &= u, \quad \eta \in R^n,\ \xi,\ u \in R\end{aligned}$$

Stabilize the origin using state feedback

View ξ as "virtual" control input to

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$

Suppose there is $\xi = \phi(\eta)$ that stabilizes the origin of

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \le -W(\eta), \quad \forall\, \eta \in D$$

$$z = \xi - \phi(\eta)$$

$$\begin{aligned}
\dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\
\dot{z} &= u - \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]
\end{aligned}$$

$$u = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] + v$$

$$\begin{aligned}
\dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\
\dot{z} &= v
\end{aligned}$$

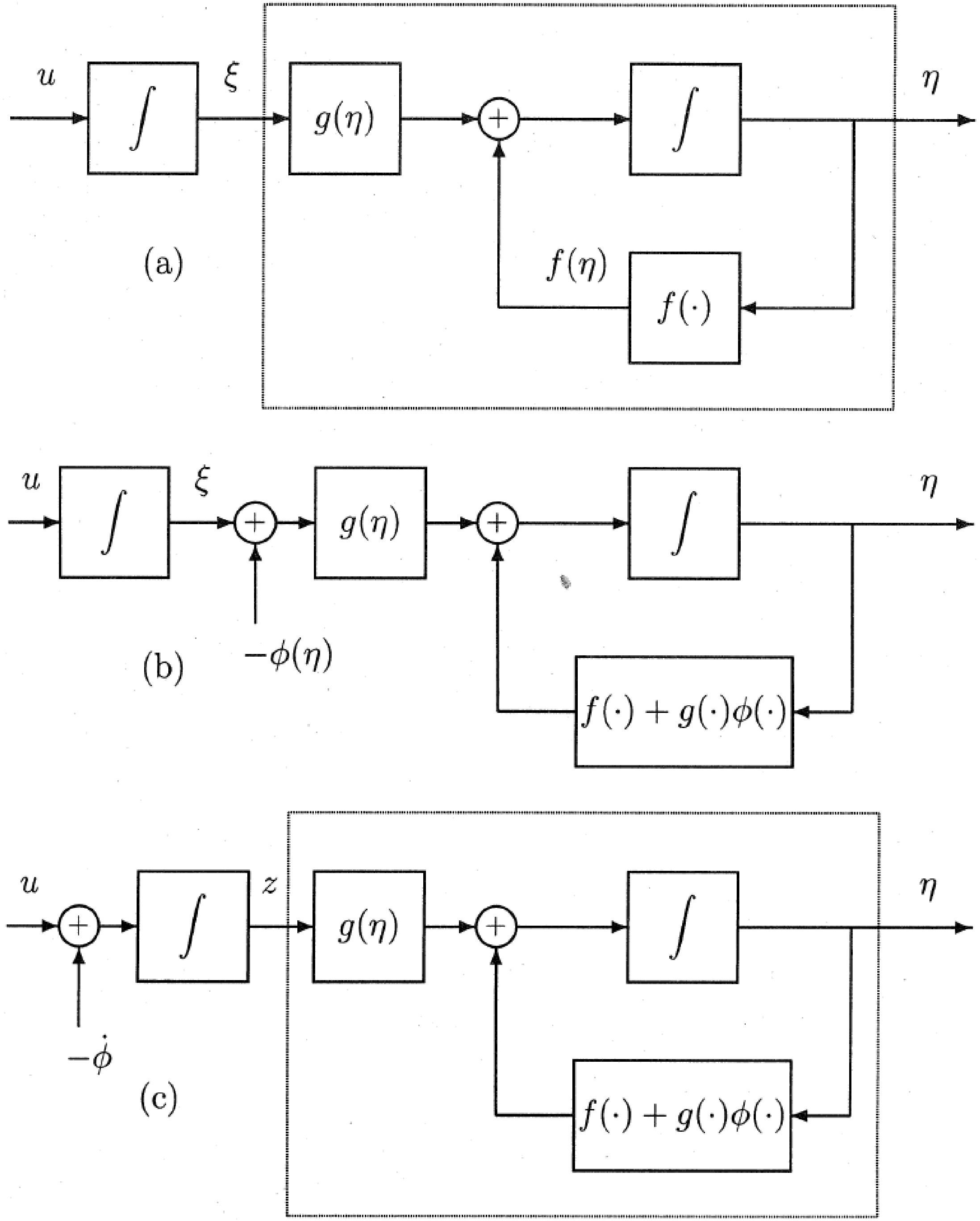
$$V_c(\eta, \xi) = V(\eta) + \frac{1}{2}z^2$$

$$\begin{aligned}\dot{V}_c &= \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta}g(\eta)z + zv \\ &\leq -W(\eta) + \frac{\partial V}{\partial \eta}g(\eta)z + zv\end{aligned}$$

$$v = -\frac{\partial V}{\partial \eta}g(\eta) - kz, \quad k > 0$$

$$\dot{V}_c \leq -W(\eta) - kz^2$$

$$u = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta}g(\eta) - k[\xi - \phi(\eta)]$$



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Example

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

$$x_2 = \phi(x_1) = -x_1^2 - x_1 \;\Rightarrow\; \dot{x}_1 = -x_1 - x_1^3$$

$$V(x_1) = \tfrac{1}{2}x_1^2 \;\Rightarrow\; \dot{V} = -x_1^2 - x_1^4, \quad \forall\, x_1 \in R$$

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$$

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1^3 + z_2 \\ \dot{z}_2 &= u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \end{aligned}$$

$$V_c(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2$$

$$\begin{aligned}\dot{V}_c &= x_1(-x_1 - x_1^3 + z_2) \\ &\quad + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)]\end{aligned}$$

$$\begin{aligned}\dot{V}_c &= -x_1^2 - x_1^4 \\ &\quad + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u]\end{aligned}$$

$$u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

$$\dot{V}_c = -x_1^2 - x_1^4 - z_2^2$$

$$\dot{V}_c = -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2$$

Example

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3$$

$$x_3 = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \stackrel{\text{def}}{=} \phi(x_1, x_2)$$

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2, \quad \dot{V} = -x_1^2 - x_1^4 - z_2^2$$

$$z_3 = x_3 - \phi(x_1, x_2)$$

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = \phi(x_1, x_2) + z_3 \\ \dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3) \end{aligned}$$

$$V_c = V + \tfrac{1}{2} z_3^2$$

$$\begin{aligned}\dot{V}_c &= \frac{\partial V}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V}{\partial x_2}(z_3 + \phi) \\ &\quad + z_3 \left[u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) \right]\end{aligned}$$

$$\begin{aligned}\dot{V}_c &= -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \\ &\quad + z_3 \left[\frac{\partial V}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u \right]\end{aligned}$$

$$u = -\frac{\partial V}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$$

$$
\begin{aligned}
\dot{\eta} &= f(\eta) + g(\eta)\xi \\
\dot{\xi} &= f_a(\eta, \xi) + g_a(\eta, \xi)u, \qquad g_a(\eta, \xi) \neq 0
\end{aligned}
$$

$$
u = \frac{1}{g_a(\eta, \xi)}[v - f_a(\eta, \xi)]
$$

$$
\begin{aligned}
\dot{\eta} &= f(\eta) + g(\eta)\xi \\
\dot{\xi} &= v
\end{aligned}
$$

$$
\begin{aligned}
u &= \phi_c(\eta, \xi) \\
&= \frac{1}{g_a(\eta, \xi)} \left\{ \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta} g(\eta) - k[\xi - \phi(\eta)] - f_a(\eta, \xi) \right\}
\end{aligned}
$$

$$
V_c(\eta, \xi) = V(\eta) + \tfrac{1}{2}[\xi - \phi(\eta)]^2
$$

Strict-Feedback Form

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x) z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1}) z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k) u
\end{aligned}$$

$$g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k$$

over the domain of interest. The recursive procedure starts with the system

$$\dot{x} = f_0(x) + g_0(x)z_1$$

where z_1 is viewed as the control input. We assume that it is possible to determine a stabilizing state feedback control law $z_1 = \phi_0(x)$, with $\phi_0(0) = 0$, and a Lyapunov function $V_0(x)$ such that

$$\frac{\partial V_0}{\partial x}[f_0(x) + g_0(x)\phi_0(x)] \leq -W(x)$$

over the domain of interest for some positive definite function $W(x)$. In many applications of backstepping, the variable x is scalar, which simplifies this stabilization problem. With $\phi_0(x)$ and $V_0(x)$ in hand, we proceed to apply backstepping in a systematic way. First, we consider the system

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x)z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \end{aligned}$$

as a special case of (14.53)–(14.54) with

$$\eta = x, \quad \xi = z_1, \quad u = z_2, \quad f = f_0, \quad g = g_0, \quad f_a = f_1, \quad g_a = g_1$$

We use (14.56) and (14.57) to obtain the stabilizing state feedback control law and the Lyapunov function as

$$\phi_1(x, z_1) = \frac{1}{g_1}\left[\frac{\partial \phi_0}{\partial x}(f_0 + g_0 z_1) - \frac{\partial V_0}{\partial x} g_0 - k_1(z_1 - \phi) - f_1\right], \qquad k_1 > 0$$

$$V_1(x, z_1) = V_0(x) + \tfrac{1}{2}[z_1 - \phi(x)]^2$$

Next, we consider the system

$$\begin{aligned} \dot{x} &= f_0(x) + g_0(x) z_1 \\ \dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2 \\ \dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3 \end{aligned}$$

as a special case of (14.53)–(14.54) with

$$\eta = \begin{bmatrix} x \\ z_1 \end{bmatrix}, \; \xi = z_2, \; u = z_3, \; f = \begin{bmatrix} f_0 + g_0 z_1 \\ f_1 \end{bmatrix}, \; g = \begin{bmatrix} 0 \\ g_1 \end{bmatrix}, \; f_a = f_2, \; g_a = g_2$$

Using (14.56) and (14.57), we obtain the stabilizing state feedback control law and the Lyapunov function as

$$\phi_2(x, z_1, z_2) = \frac{1}{g_2}\left[\frac{\partial \phi_1}{\partial x}(f_0 + g_0 z_1) + \frac{\partial \phi_1}{\partial z_1}(f_1 + g_1 z_2) - \frac{\partial V_1}{\partial z_1} g_1 - k_2(z_2 - \phi_1) - f_2\right]$$

for some $k_2 > 0$ and

$$V_2(x, z_1, z_2) = V_1(x, z_1) + \tfrac{1}{2}[z_2 - \phi_2(x, z_1)]^2$$

This process is repeated k times to obtain the overall stabilizing state feedback control law $u = \phi_k(x, z_1, \ldots, z_k)$ and the Lyapunov function $V_k(x, z_1, \ldots, z_k)$.