

Chapter 1

Introduction

Examples of Nonlinear Systems

Nonlinear State Model

$$\begin{aligned}\dot{x}_1 &= f_1(t, x_1, \dots, x_n, u_1, \dots, u_p) \\ \dot{x}_2 &= f_2(t, x_1, \dots, x_n, u_1, \dots, u_p) \\ &\vdots \\ \dot{x}_n &= f_n(t, x_1, \dots, x_n, u_1, \dots, u_p)\end{aligned}$$

\dot{x}_i denotes the derivative of x_i with respect to the time variable t

u_1, u_2, \dots, u_p are input variables

x_1, x_2, \dots, x_n the state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ \vdots \\ f_n(t, x, u) \end{bmatrix}$$

$$\dot{x} = f(t, x, u)$$

$$\dot{x} = f(t, x, u)$$

$$y = h(t, x, u)$$

x is the state, u is the input
 y is the output (q -dimensional vector)

Special Cases:

Linear systems:

$$\dot{x} = A(t)x + B(t)u$$

$$y = C(t)x + D(t)u$$

Unforced state equation:

$$\dot{x} = f(t, x)$$

Results from $\dot{x} = f(t, x, u)$ with $u = \gamma(t, x)$

Autonomous System:

$$\dot{x} = f(x)$$

Time-Invariant System:

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

A time-invariant state model has a time-invariance property with respect to shifting the initial time from t_0 to $t_0 + a$, provided the input waveform is applied from $t_0 + a$ rather than t_0

Equilibrium Points

A point $x = x^*$ in the state space is said to be an equilibrium point of $\dot{x} = f(t, x)$ if

$$x(t_0) = x^* \Rightarrow x(t) \equiv x^*, \quad \forall t \geq t_0$$

For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real solutions of the equation

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points

A linear system $\dot{x} = Ax$ can have an isolated equilibrium point at $x = 0$ (if A is nonsingular) or a continuum of equilibrium points in the null space of A (if A is singular)

It cannot have multiple isolated equilibrium points, for if x_a and x_b are two equilibrium points, then by linearity any point on the line $\alpha x_a + (1 - \alpha)x_b$ connecting x_a and x_b will be an equilibrium point

A nonlinear state equation can have multiple isolated equilibrium points. For example, the state equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 - bx_2$$

has equilibrium points at $(x_1 = n\pi, x_2 = 0)$ for $n = 0, \pm 1, \pm 2, \dots$

Linearization

A common engineering practice in analyzing a nonlinear system is to linearize it about some nominal operating point and analyze the resulting linear model

What are the limitations of linearization?

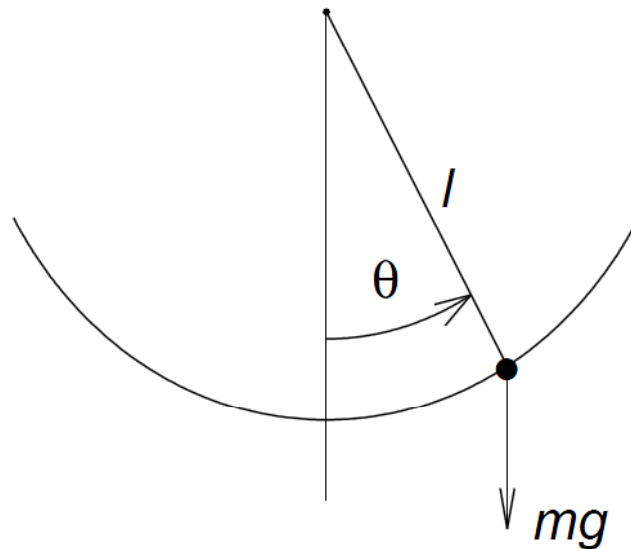
- Since linearization is an approximation in the neighborhood of an operating point, it can only predict the “local” behavior of the nonlinear system in the vicinity of that point. It cannot predict the “nonlocal” or “global” behavior
- There are “essentially nonlinear phenomena” that can take place only in the presence of nonlinearity

Nonlinear Phenomena

- Finite escape time
- Multiple isolated equilibrium points
- Limit cycles
- Subharmonic, harmonic, or almost-periodic oscillations
- Chaos
- Multiple modes of behavior

Nonlinear Systems Examples

- Pendulum Equation



$$ml\ddot{\theta} = -mg \sin \theta - kl\dot{\theta}$$

$$x_1 = \theta, \quad x_2 = \dot{\theta}$$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2\end{aligned}$$

Equilibrium Points:

$$\begin{aligned}0 &= x_2 \\ 0 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2\end{aligned}$$

$$(n\pi, 0) \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Nontrivial equilibrium points at $(0, 0)$ and $(\pi, 0)$

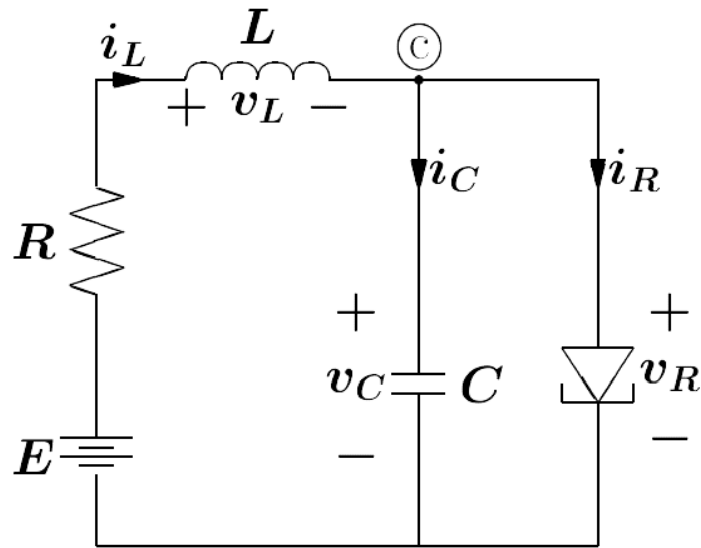
Pendulum without friction:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l} \sin x_1\end{aligned}$$

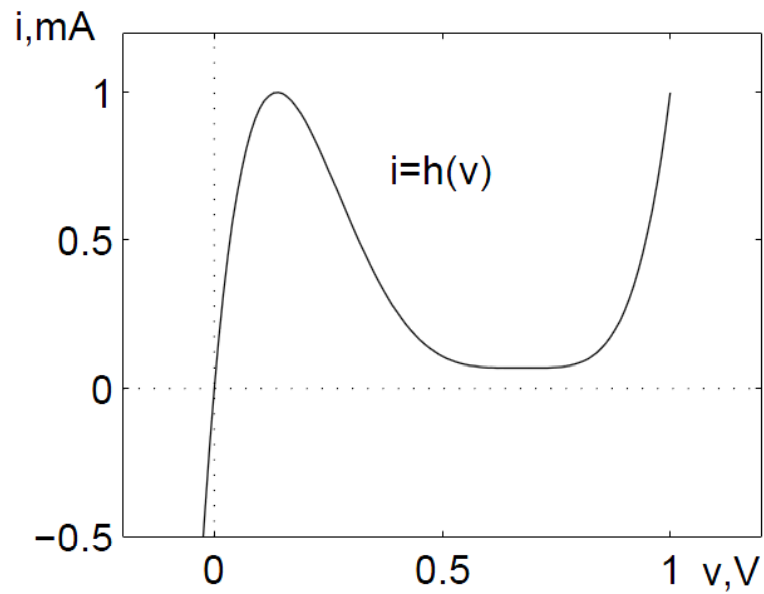
Pendulum with torque input:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l} \sin x_1 - \frac{k}{m} x_2 + \frac{1}{ml^2} T\end{aligned}$$

Tunnel-Diode Circuit



(a)



(b)

$$i_C = C \frac{dv_C}{dt}, \quad v_L = L \frac{di_L}{dt}$$

$$x_1 = v_C, \quad x_2 = i_L, \quad u = E$$

$$i_C + i_R - i_L = 0 \Rightarrow i_C = -h(x_1) + x_2$$

$$v_C - E + Ri_L + v_L = 0 \Rightarrow v_L = -x_1 - Rx_2 + u$$

$$\dot{x}_1 = \frac{1}{C} [-h(x_1) + x_2]$$

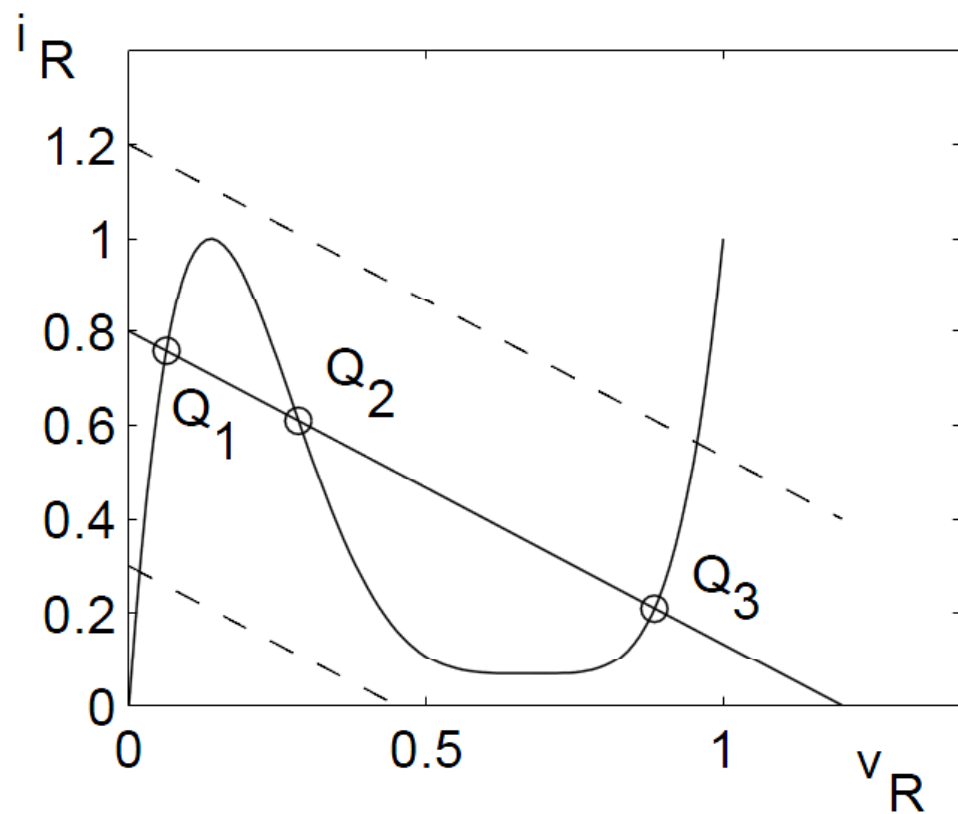
$$\dot{x}_2 = \frac{1}{L} [-x_1 - Rx_2 + u]$$

Equilibrium Points:

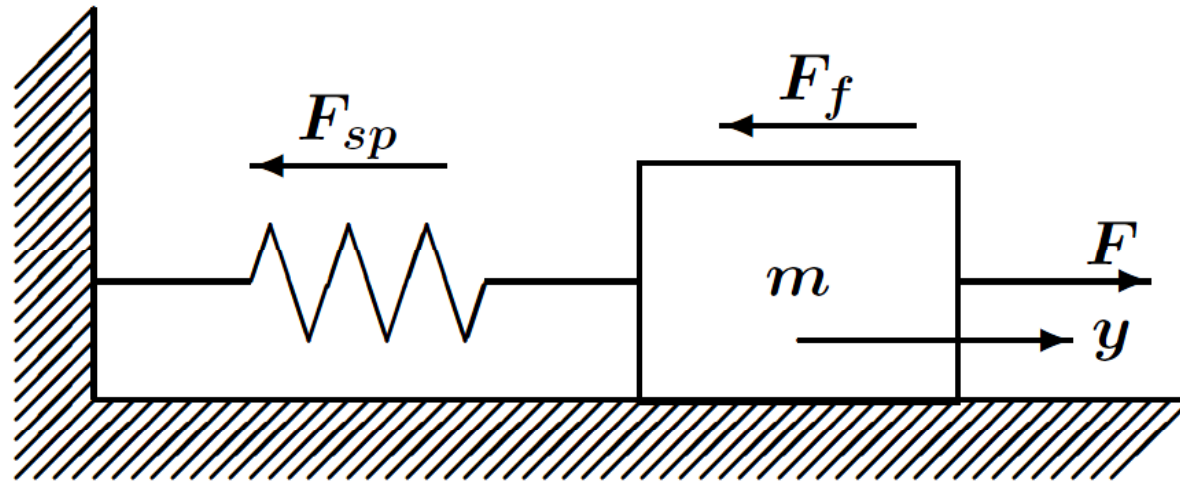
$$0 = -h(x_1) + x_2$$

$$0 = -x_1 - Rx_2 + u$$

$$h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$



Mass-Spring System



$$m\ddot{y} + F_f + F_{sp} = F$$

Sources of nonlinearity:

- Nonlinear spring restoring force $F_{sp} = g(y)$
- Static or Coulomb friction

$$F_{sp} = g(y)$$

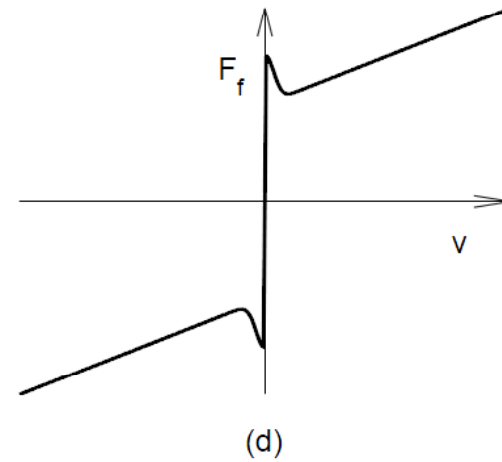
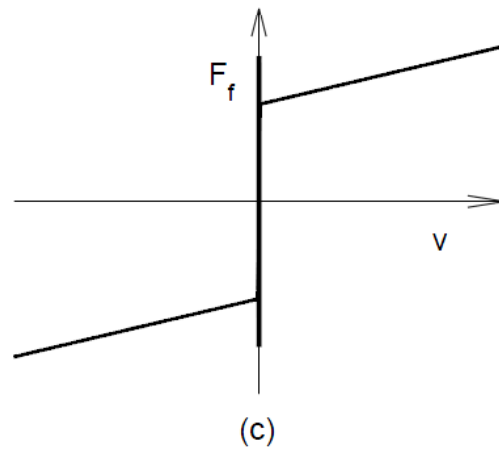
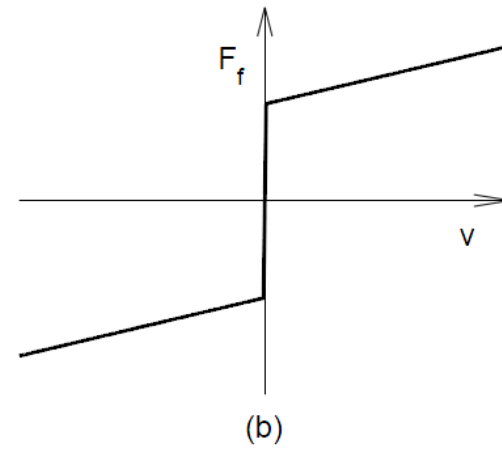
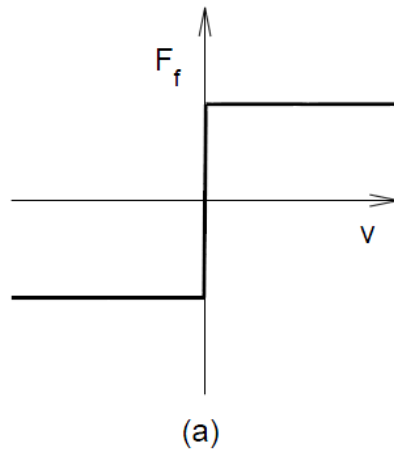
$$g(y) = k(1 - a^2 y^2)y, \quad |ay| < 1 \quad (\text{softening spring})$$

$$g(y) = k(1 + a^2 y^2)y \quad (\text{hardening spring})$$

F_f may have components due to static, Coulomb, and viscous friction

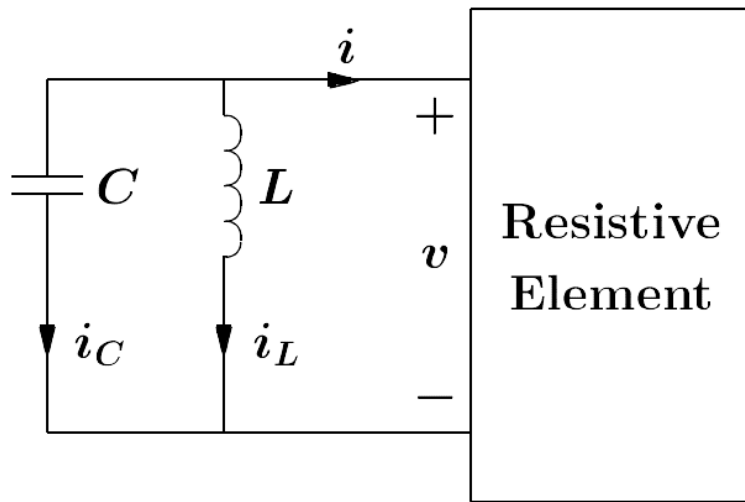
When the mass is at rest, there is a static friction force F_s that acts parallel to the surface and is limited to $\pm \mu_s mg$ ($0 < \mu_s < 1$). F_s takes whatever value, between its limits, to keep the mass at rest

Once motion has started, the resistive force F_f is modeled as a function of the sliding velocity $v = \dot{y}$

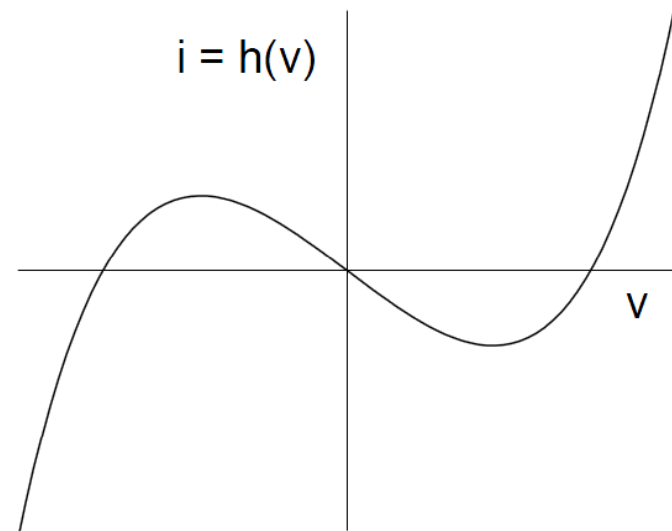


(a) Coulomb friction; (b) Coulomb plus linear viscous friction; (c) static, Coulomb, and linear viscous friction; (d) static, Coulomb, and linear viscous friction—Stribeck effect

Negative-Resistance Oscillator



(a)



(b)

$$h(0) = 0, \quad h'(0) < 0$$

$$h(v) \rightarrow \infty \text{ as } v \rightarrow \infty, \text{ and } h(v) \rightarrow -\infty \text{ as } v \rightarrow -\infty$$

$$i_C + i_L + i = 0$$

$$C \frac{dv}{dt} + \frac{1}{L} \int_{-\infty}^t v(s) ds + h(v) = 0$$

Differentiating with respect to t and multiplying by L :

$$CL \frac{d^2v}{dt^2} + v + Lh'(v) \frac{dv}{dt} = 0$$

$$\tau = t / \sqrt{CL}$$

$$\frac{dv}{d\tau} = \sqrt{CL} \frac{dv}{dt}, \quad \frac{d^2v}{d\tau^2} = CL \frac{d^2v}{dt^2}$$

Denote the derivative of v with respect to τ by \dot{v}

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0, \quad \varepsilon = \sqrt{L/C}$$

Special case: Van der Pol equation

$$h(v) = -v + \frac{1}{3}v^3$$

$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0$$

State model: $x_1 = v, \quad x_2 = \dot{v}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - \varepsilon h'(x_1)x_2$$

Another State Model: $z_1 = i_L, \quad z_2 = v_C$

$$\begin{aligned}\dot{z}_1 &= \frac{1}{\varepsilon} z_2 \\ \dot{z}_2 &= -\varepsilon [z_1 + h(z_2)]\end{aligned}$$

Change of variables: $z = T(x)$

$$x_1 = v = z_2$$

$$\begin{aligned}x_2 &= \frac{dv}{d\tau} = \sqrt{CL} \frac{dv}{dt} = \sqrt{\frac{L}{C}} [-i_L - h(v_C)] \\ &= \varepsilon [-z_1 - h(z_2)]\end{aligned}$$

$$T(x) = \begin{bmatrix} -h(x_1) - \frac{1}{\varepsilon} x_2 \\ x_1 \end{bmatrix}, \quad T^{-1}(z) = \begin{bmatrix} z_2 \\ -\varepsilon z_1 - \varepsilon h(z_2) \end{bmatrix}$$

Adaptive Control

$$\textit{Plant} : \quad \dot{y}_p = a_p y_p + k_p u$$

$$\textit{Reference Model} : \quad \dot{y}_m = a_m y_m + k_m r$$

$$u(t) = \theta_1^* r(t) + \theta_2^* y_p(t)$$

$$\theta_1^* = \frac{k_m}{k_p} \quad \text{and} \quad \theta_2^* = \frac{a_m - a_p}{k_p}$$

When a_p and k_p are unknown, we may use

$$u(t) = \theta_1(t)r(t) + \theta_2(t)y_p(t)$$

where $\theta_1(t)$ and $\theta_2(t)$ are adjusted on-line

Adaptive Law (gradient algorithm):

$$\begin{aligned}\dot{\theta}_1 &= -\gamma(y_p - y_m)r \\ \dot{\theta}_2 &= -\gamma(y_p - y_m)y_p, \quad \gamma > 0\end{aligned}$$

State Variables: $e_o = y_p - y_m$, $\phi_1 = \theta_1 - \theta_1^*$, $\phi_2 = \theta_2 - \theta_2^*$

$$\dot{y}_m = a_p y_m + k_p(\theta_1^* r + \theta_2^* y_m)$$

$$\dot{y}_p = a_p y_p + k_p(\theta_1 r + \theta_2 y_p)$$

$$\dot{e}_o = a_p e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 y_p - \theta_2^* y_m)$$

$$= \dots + k_p[\theta_2^* y_p - \theta_2^* y_p]$$

$$= (a_p + k_p \theta_2^*)e_o + k_p(\theta_1 - \theta_1^*)r + k_p(\theta_2 - \theta_2^*)y_p$$

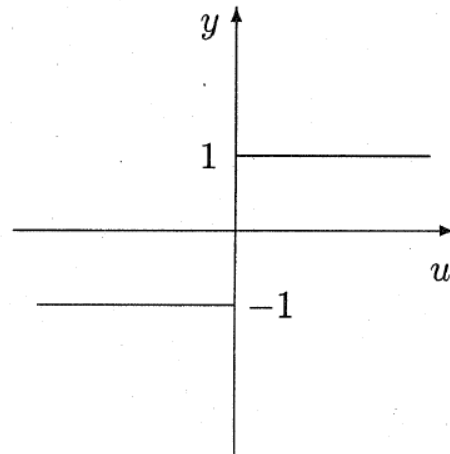
Closed-Loop System:

$$\dot{e}_o = a_m e_o + k_p \phi_1 r(t) + k_p \phi_2 [e_o + y_m(t)]$$

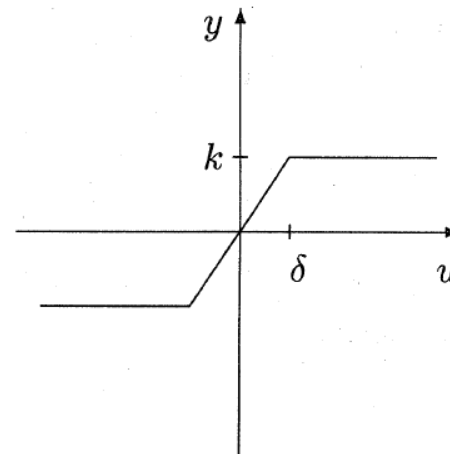
$$\dot{\phi}_1 = -\gamma e_o r(t)$$

$$\dot{\phi}_2 = -\gamma e_o [e_o + y_m(t)]$$

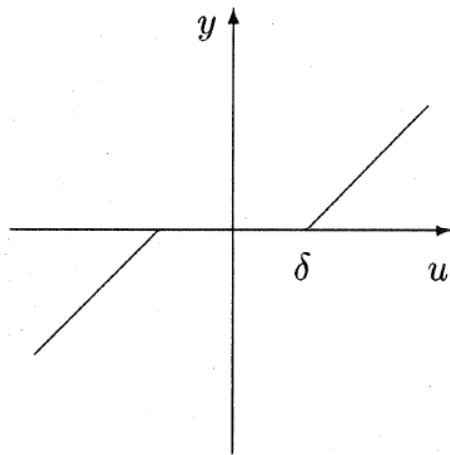
Common Nonlinearities



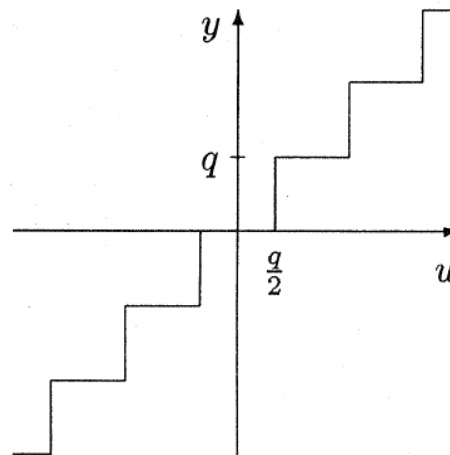
(a) Relay



(b) Saturation



(c) Dead zone



(d) Quantization