

Chapter 2

Second-Order Systems

$$\dot{x}_1 = f_1(x_1, x_2) = f_1(x)$$

$$\dot{x}_2 = f_2(x_1, x_2) = f_2(x)$$

Let $x(t) = (x_1(t), x_2(t))$ be a solution that starts at initial state $x_0 = (x_{10}, x_{20})$. The locus in the x_1 – x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0 . This curve is called a *trajectory* or *orbit*

The x_1 – x_2 plane is called the *state plane* or *phase plane*

The family of all trajectories is called the *phase portrait*

The *vector field* $f(x) = (f_1(x), f_2(x))$ is tangent to the trajectory at point x because

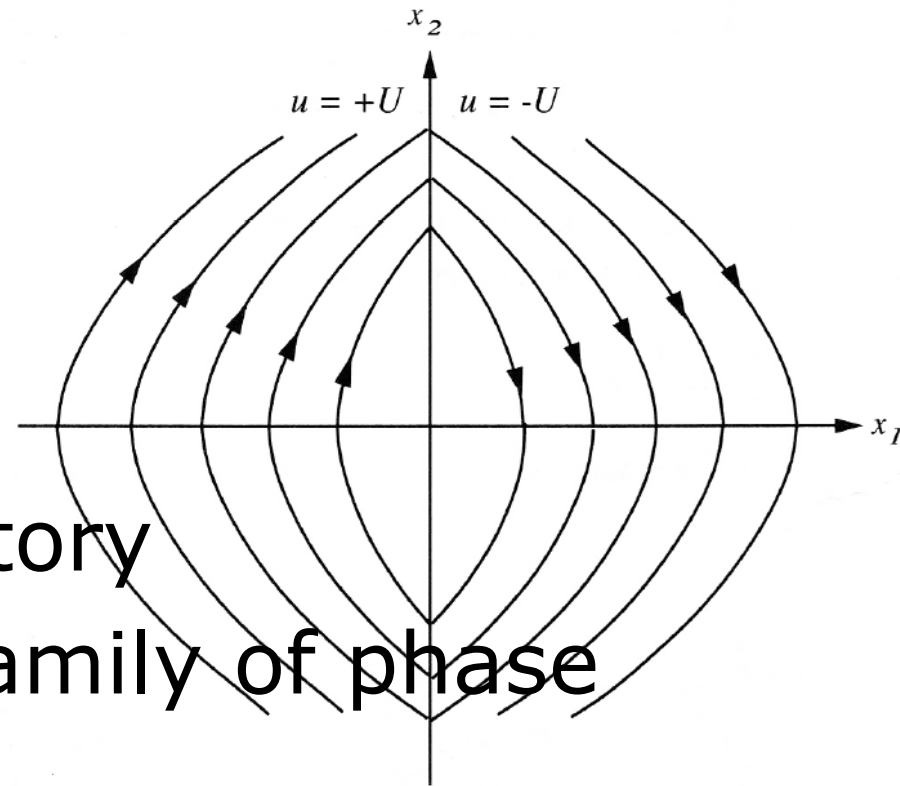
$$\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}$$

Phase Portrait

- Graphical study of second order systems

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$



- Phase plane trajectory
- Phase portrait: a family of phase plane trajectories

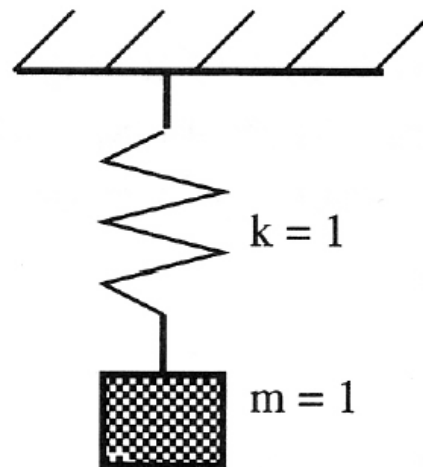
Phase portrait of a mass-spring system

$$\ddot{x} + x = 0$$

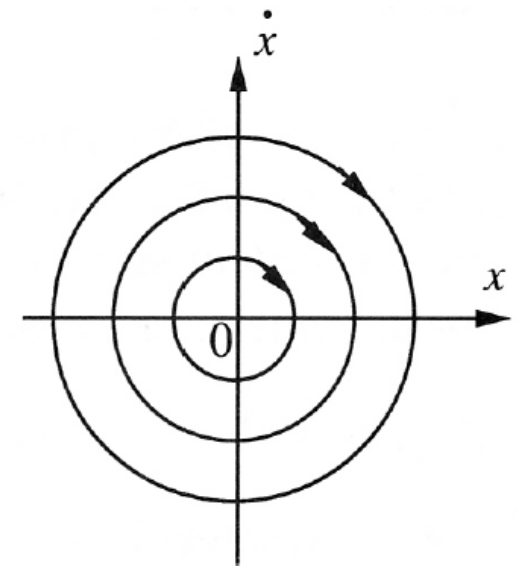
$$x(t) = x_o \cos t$$

$$\dot{x}(t) = -x_o \sin t$$

$$x^2 + \dot{x}^2 = x_o^2$$



(a)



(b)

Figure 2.1 : A mass-spring system and its phase portrait

A major class of second-order system

$$\ddot{x} + f(x, \dot{x}) = 0$$

with $x_1 = x$ and $x_2 = \dot{x}$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -f(x_1, x_2)$$

Singular Points

- Singular point: equilibrium point in the phase plane
- Equilibrium point: a point where the system states can stay forever

solving $\dot{\mathbf{x}} = \mathbf{0}$ for the state \mathbf{x} , *i.e.*, from solving

$$f_1(x_1, x_2) = 0 \quad f_2(x_1, x_2) = 0$$

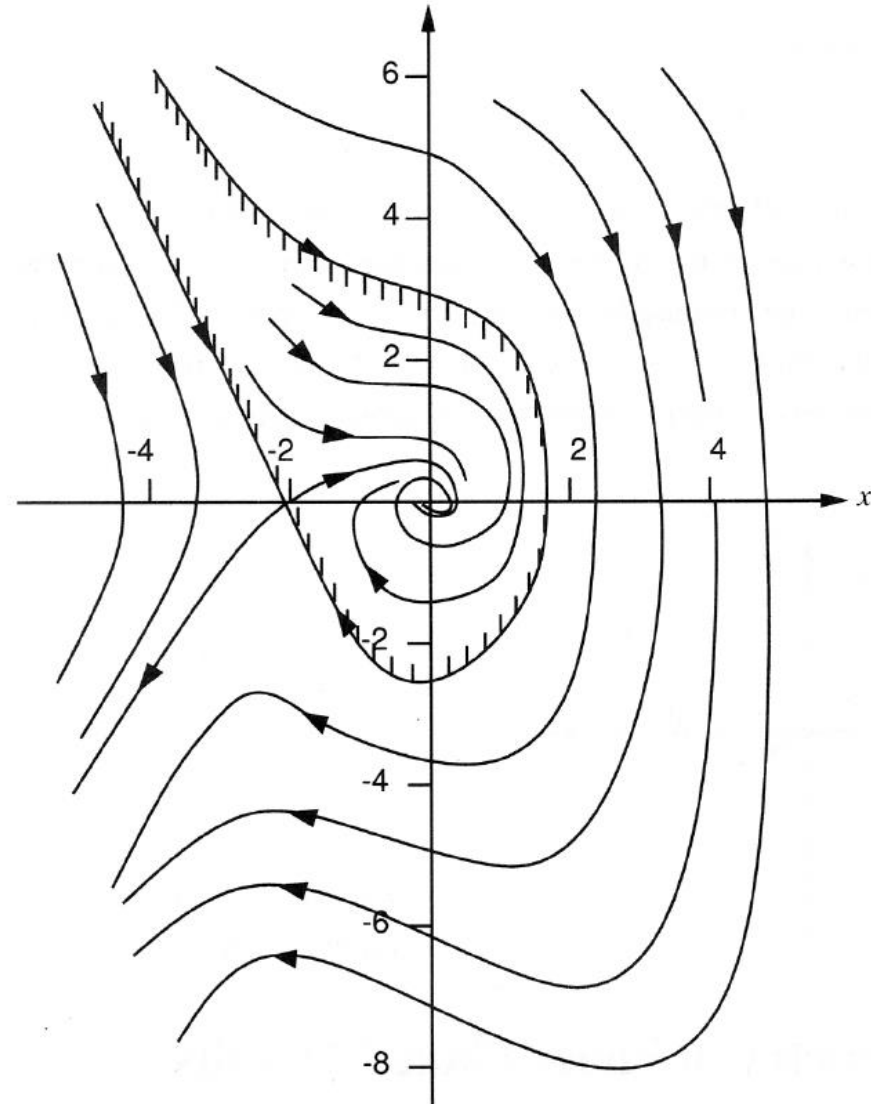
- Linear system: usually only one singular point
- Nonlinear system: more than one

A nonlinear second-order system

$$\ddot{x} + 0.5 \dot{x} + 2x + x^2 = 0$$

$$\dot{x} = 0, \ddot{x} = 0 \rightarrow 2x + x^2 = 0$$

$$\rightarrow x = 0, -2$$



Phase Portrait MATLAB(1)

```
clear
clf
Tfinal=20;
Td=0.01;
t=0:Td:Tfinal;
N=length(t);
x(1,1)=-2;x(2,1)=2;
for k=2:N
    x(1,k)=x(1,k-1)+Td*x(2,k-1);
    x(2,k)=x(2,k-1)+Td*(-0.5*x(2,k-1)-2*x(1,k-1)-x(1,k-1)^2);
end
plot(x(1,:),x(2,:))
axis([-2 2 -2 2]);
hold on
```

$$x_1 = x, x_2 = \dot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -0.5x_2 - 2x_1 + x_1^2$$

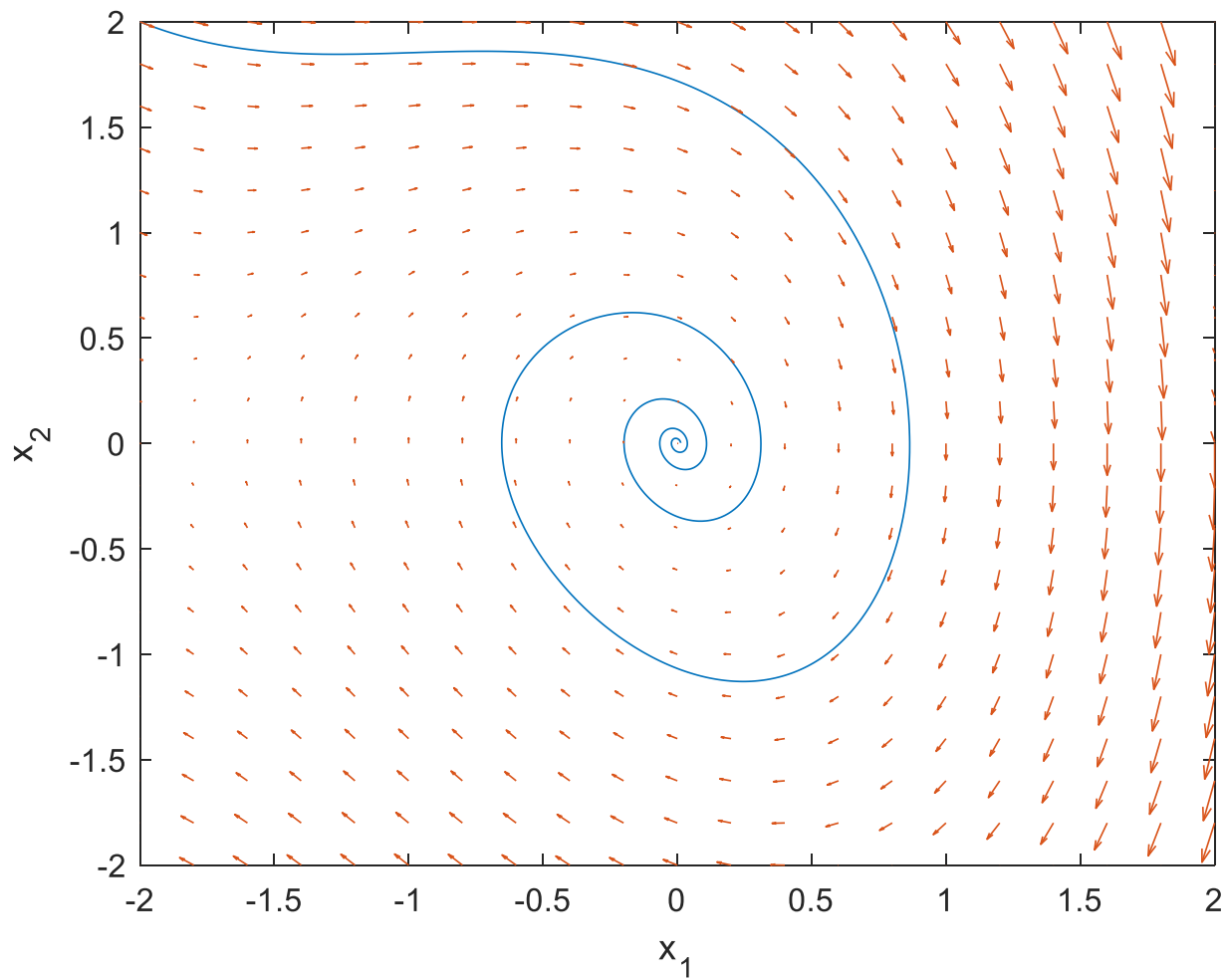
Phase Portrait MATLAB(2)

```
[x1, x2] = meshgrid(-2:0.2:2, -2:0.2:2);  
x1dot = x2;  
x2dot = -0.5*x2-2*x1-x1.^2;  
quiver(x1,x2,x1dot,x2dot)  
xlabel('x_1')  
ylabel('x_2')
```

$$x_1 = x, x_2 = \dot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -0.5x_2 - 2x_1 + x_1^2$$



Singular Point

- The slope of the phase trajectories

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

With the functions f_1 and f_2 assumed to be single valued, there is usually a definite value for this slope at any given point in phase plane. This implies that the phase trajectories will not intersect. At singular points, however, the value of the slope is $0/0$, *i.e.*, the slope is indeterminate. Many trajectories may intersect at such points, as seen from Figure 2.2. This indeterminacy of the slope accounts for the adjective "singular".

Constructing Phase Portraits

- Analytical Method

$$g(x_1, x_2, c) = 0$$

The first technique involves solving equations (2.1) for x_1 and x_2 as functions of time t , *i.e.*,

$$x_1(t) = g_1(t) \quad x_2(t) = g_2(t)$$

and then eliminating time t from these equations, leading to a functional relation in the form of (2.6). This technique was already illustrated in Example 2.1.

Constructing Phase Portraits

- Analytical Method

The second technique, on the other hand, involves directly eliminating the time variable, by noting that

$$\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

and then solving this equation for a functional relation between x_1 and x_2 . Let us use

Mass-spring system

$$\ddot{x} + x = 0$$

$$x_1 = x, x_2 = \dot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x} = -x = -x_1$$

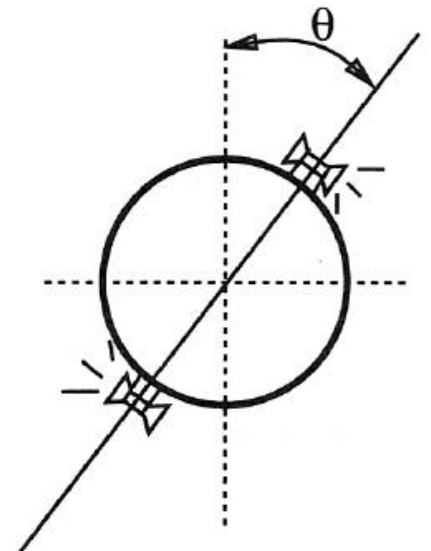
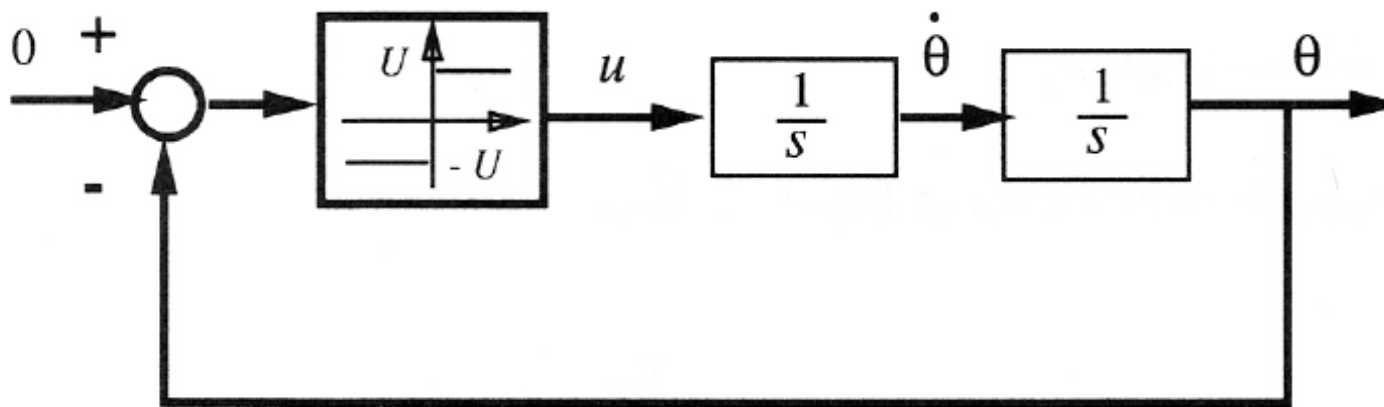
$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} \rightarrow x_2 \frac{dx_2}{dx_1} + x_1 = 0$$

$$\rightarrow \int x_2 \frac{dx_2}{dx_1} dx_1 + \int x_1 dx_1 = C \rightarrow x_2^2 + x_1^2 = x_0^2$$

A satellite control system

$$\ddot{\theta} = u$$

$$u(t) = \begin{cases} -U & \text{if } \theta > 0 \\ U & \text{if } \theta < 0 \end{cases}$$

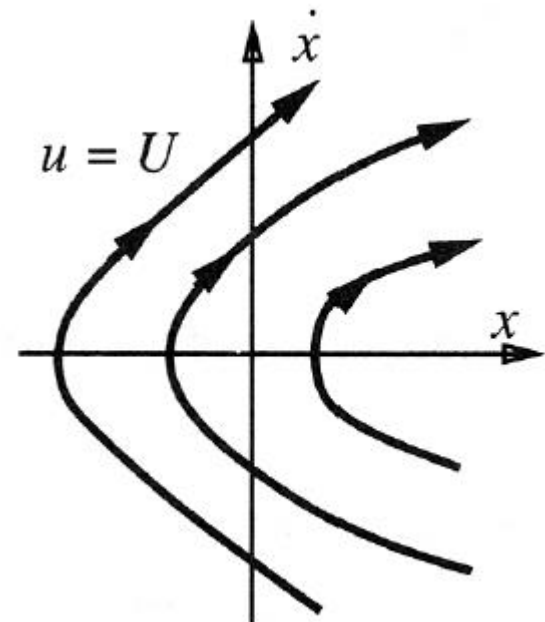


A satellite control system

$$\theta < 0$$

$$\ddot{\theta} = U \rightarrow \ddot{\theta} d\theta = \frac{d\dot{\theta}}{dt} d\theta = U d\theta \rightarrow \dot{\theta} d\dot{\theta} = U d\theta$$

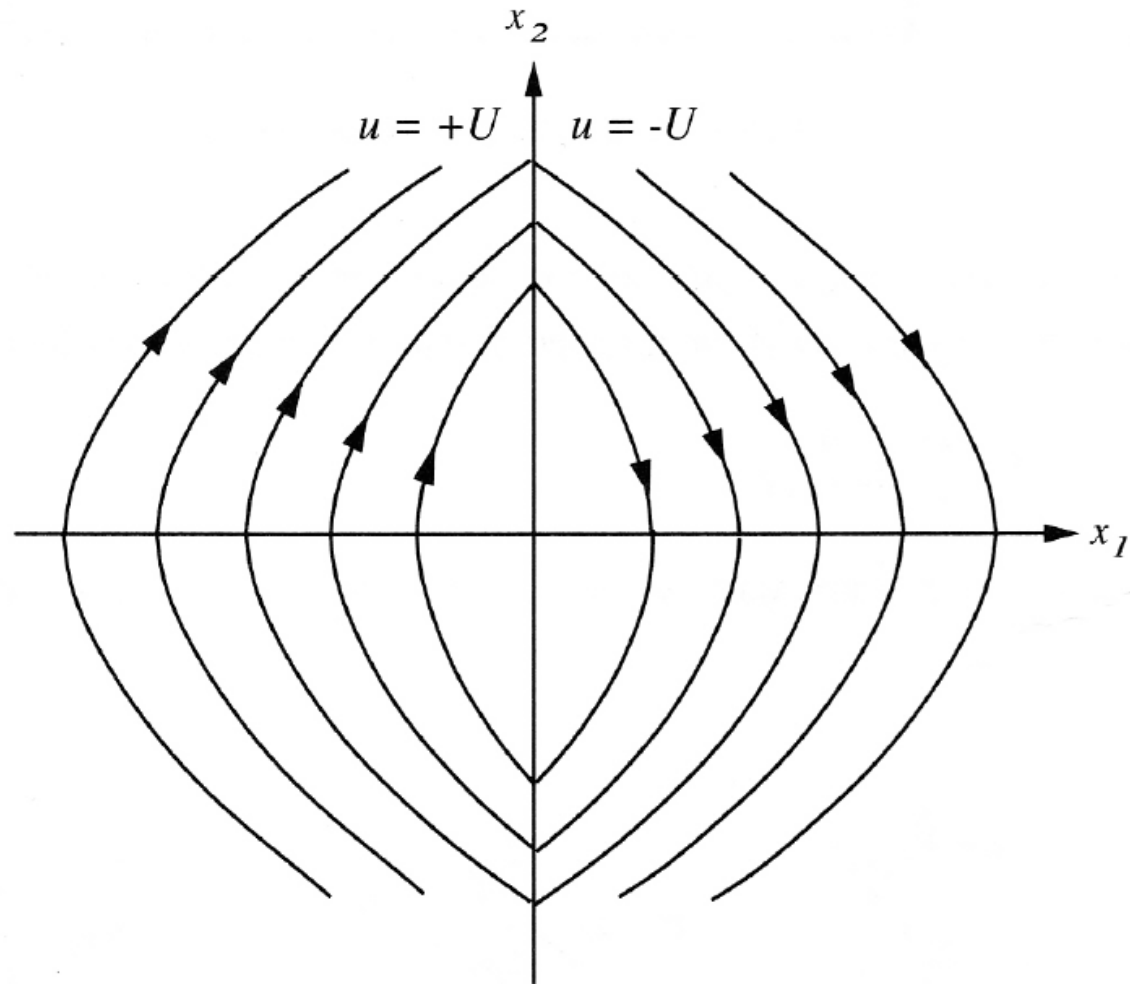
$$\rightarrow \frac{1}{2} \dot{\theta}^2 = U\theta + c_1$$



A satellite control system

$$\theta > 0$$

$$\frac{1}{2} \dot{\theta}^2 = -U\theta + c_1$$



The Method of Isoclines

locus of the points with a given tangent slope.
defined to be

$$\frac{dx_2}{dx_1} = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \alpha$$

This is to say that points on the curve

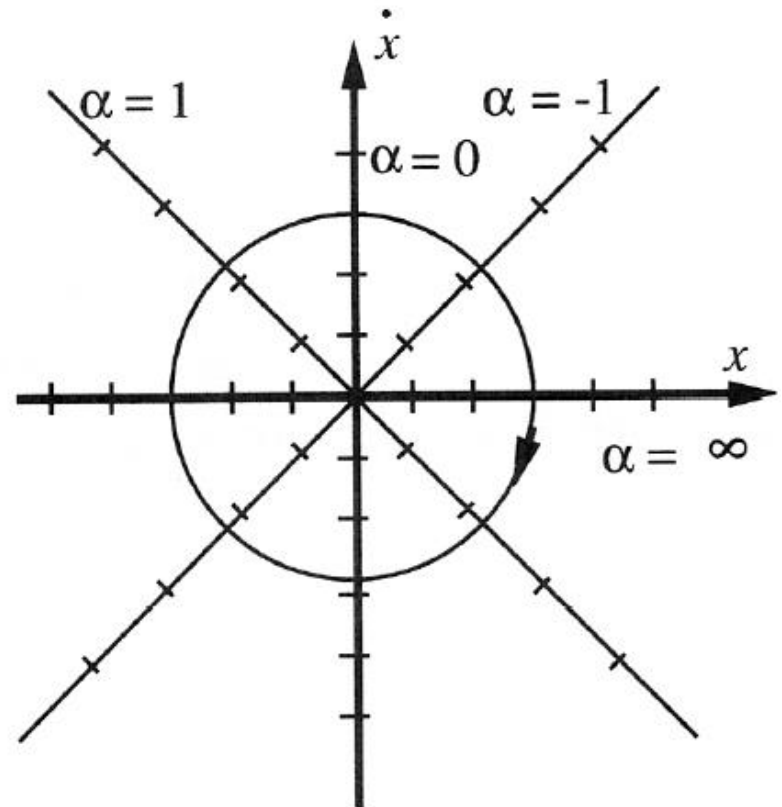
$$f_2(x_1, x_2) = \alpha f_1(x_1, x_2)$$

all have the same tangent slope α .

The Method of Isoclines

- Mass-spring system

$$\frac{dx_2}{dx_1} = \frac{-x_1}{x_2} = \alpha \rightarrow x_1 + \alpha x_2 = 0$$



The Van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

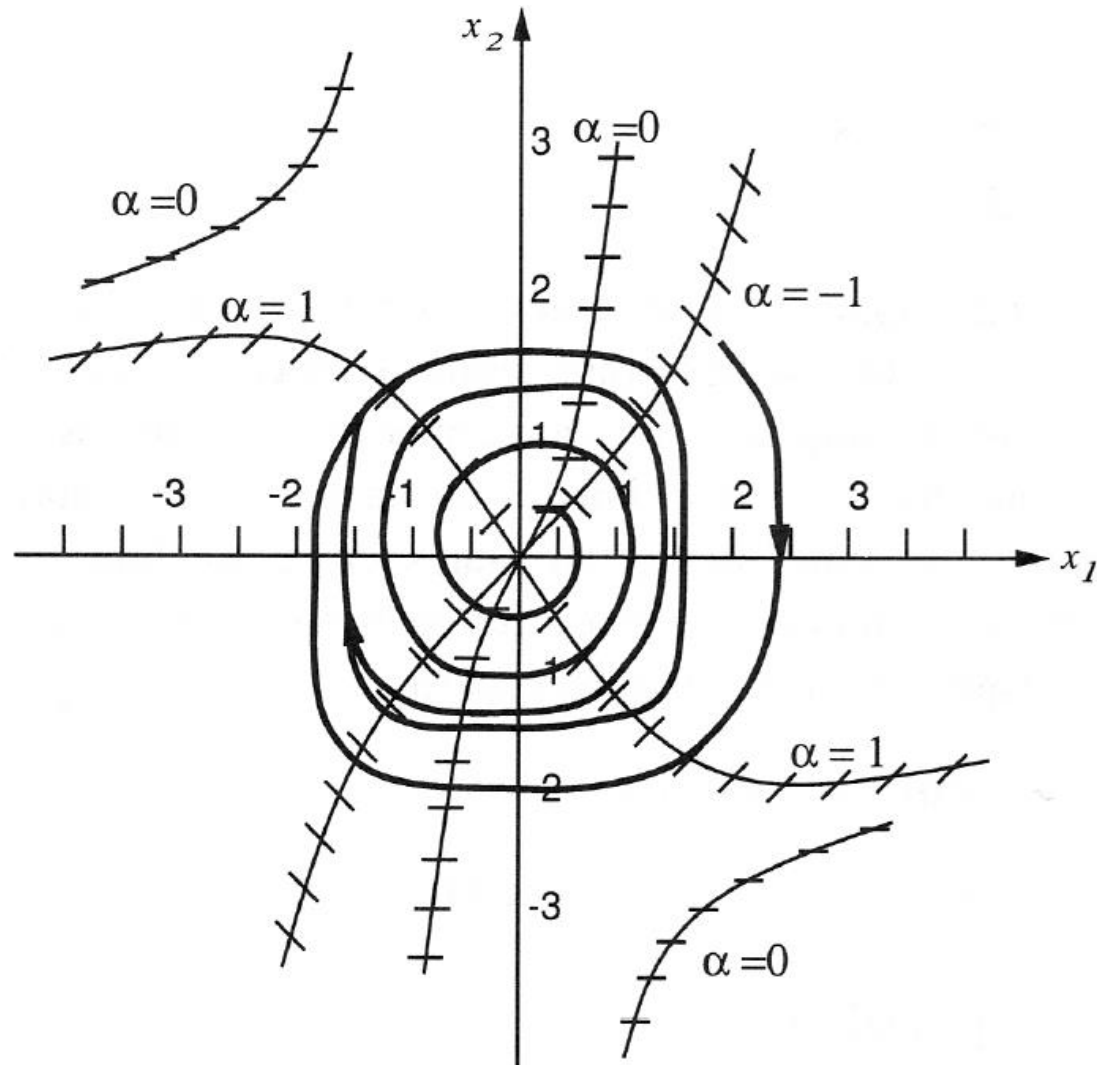
$$\frac{d\dot{x}}{dt} dx + \mu(x^2 - 1)\dot{x}dx + xdx = 0$$

$$\dot{x}d\dot{x} + \mu(x^2 - 1)\dot{x}dx + xdx = 0$$

$$\dot{x} \frac{d\dot{x}}{dx} + \mu(x^2 - 1)\dot{x} + x = 0 \rightarrow \frac{d\dot{x}}{dx} = \frac{-\mu(x^2 - 1)\dot{x} - x}{\dot{x}} = \alpha$$

$$\dot{x} = \frac{x}{(\mu - \mu x^2) - \alpha}$$

The Van der Pol equation



Phase Portrait MATLAB(1)

```
clear
clf
mu=1;
Tfinal=20;
Td=0.01;
t=0:Td:Tfinal;
N=length(t);
x(1,1)=-0.1;x(2,1)=0.1;
for k=2:N
    x(1,k)=x(1,k-1)+Td*x(2,k-1);
    x(2,k)=x(2,k-1)+Td*(-mu*(x(1,k-1)^2-1)*x(2,k-1)-x(1,k-1));
end
plot(x(1,:),x(2,:))
axis([-3 3 -3 3]);
hold on
```

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$$x_1 = x, x_2 = \dot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\mu(x_1^2 - 1)x_2 - x_1$$

Phase Portrait MATLAB(2)

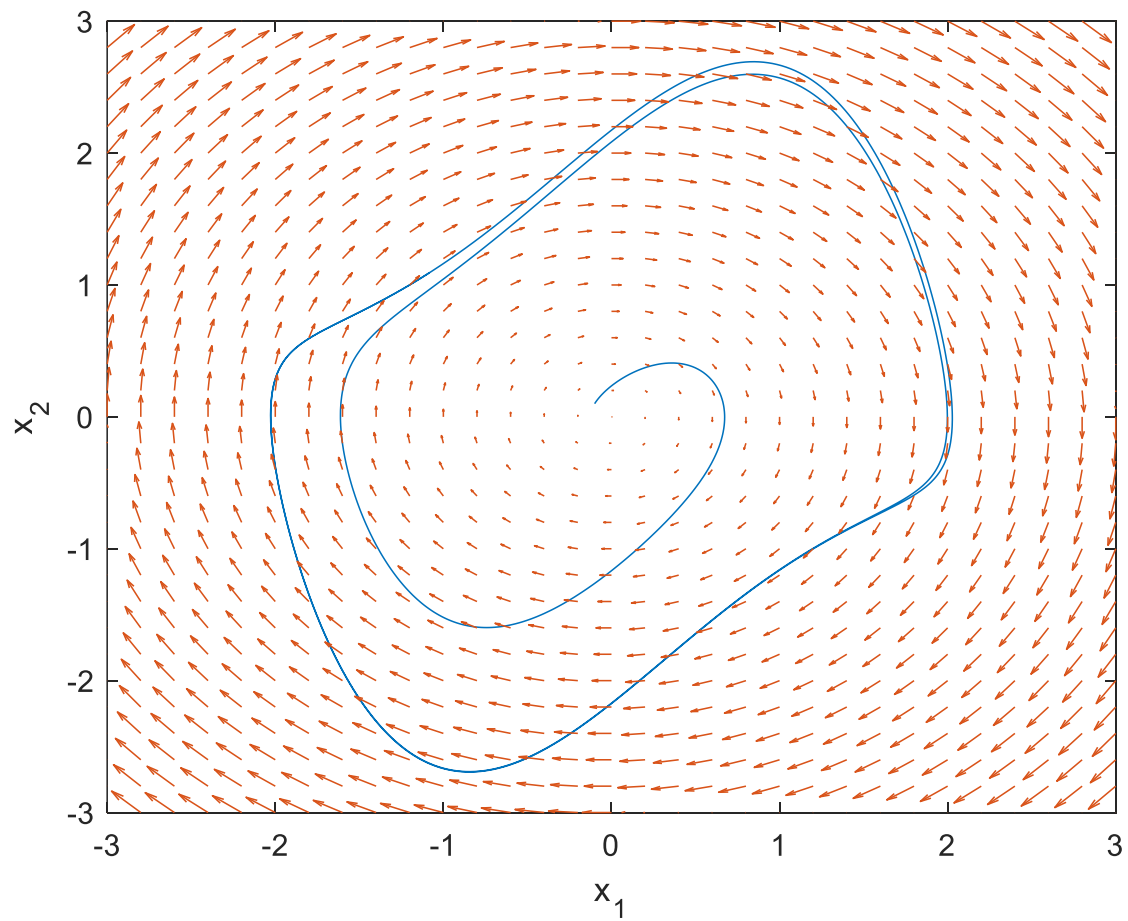
```
[x1, x2] = meshgrid(-3:0.2:3, -3:0.2:3);  
x1dot = x2;  
x2dot = -mu*(x1^2-1)*x2-x1;  
quiver(x1,x2,x1dot,x2dot)  
xlabel('x_1')  
ylabel('x_2')
```

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

$$x_1 = x, x_2 = \dot{x}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\mu(x_1^2 - 1)x_2 - x_1$$



Phase Plane Analysis of Linear Systems

- Linear 2nd order system

$$\dot{x}_1 = ax_1 + bx_2$$

$$\dot{x}_2 = cx_1 + dx_2$$

$$\dot{x} = Ax$$

Linear 2nd Order System

$$\dot{x} = Ax \rightarrow x(t) = e^{At} x(0)$$

$$\dot{z} = J_r z, J_r : \text{Jordan form of } A$$

$$A = MJ_r M^{-1}$$

$$z(t) = e^{J_r t} z(0)$$

Linear 2nd Order System

$$\mathbf{J}_r : \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix},$$

$$\begin{bmatrix} \alpha + j\beta & 0 \\ 0 & \alpha - j\beta \end{bmatrix} \rightarrow \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

Case 1. Both eigenvalues are real: $\lambda_1 \neq \lambda_2 \neq 0$

$$M = [v_1, v_2]$$

v_1 & v_2 are the real eigenvectors associated with λ_1 & λ_2

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2$$

$$z_1(t) = z_{10} e^{\lambda_1 t}, \quad z_2(t) = z_{20} e^{\lambda_2 t}$$

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \quad c = z_{20} / (z_{10})^{\lambda_2/\lambda_1}$$

The shape of the phase portrait depends on the signs of λ_1 and λ_2

$$\lambda_2 < \lambda_1 < 0$$

$e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero as $t \rightarrow \infty$

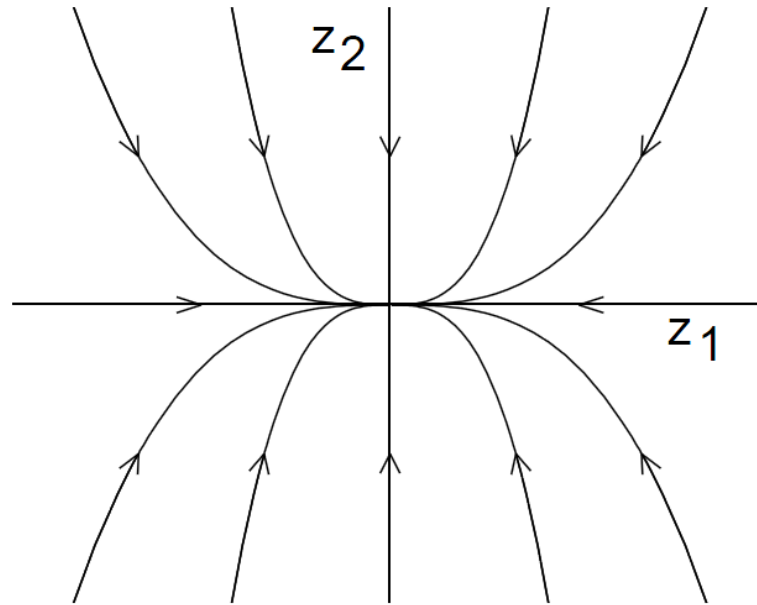
$e^{\lambda_2 t}$ tends to zero faster than $e^{\lambda_1 t}$

Call λ_2 the fast eigenvalue (v_2 the fast eigenvector) and λ_1 the slow eigenvalue (v_1 the slow eigenvector)

The trajectory tends to the origin along the curve

$$z_2 = cz_1^{\lambda_2/\lambda_1} \text{ with } \lambda_2/\lambda_1 > 1$$

$$\frac{dz_2}{dz_1} = c \frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]}$$

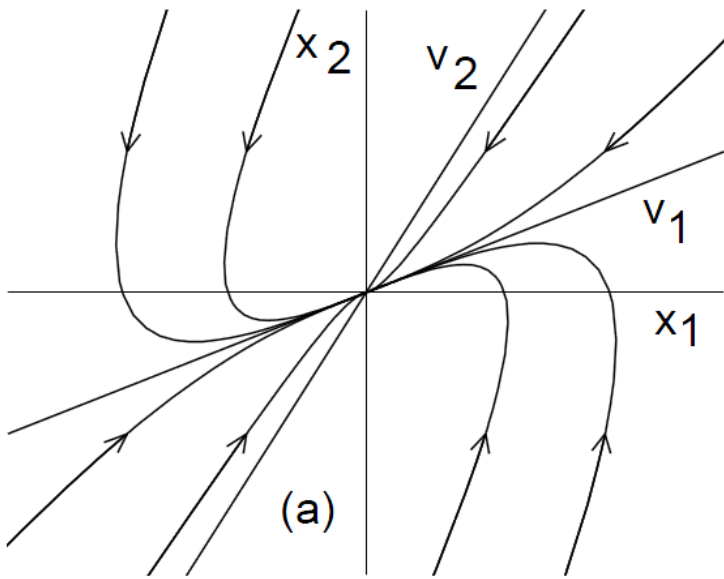


Stable Node

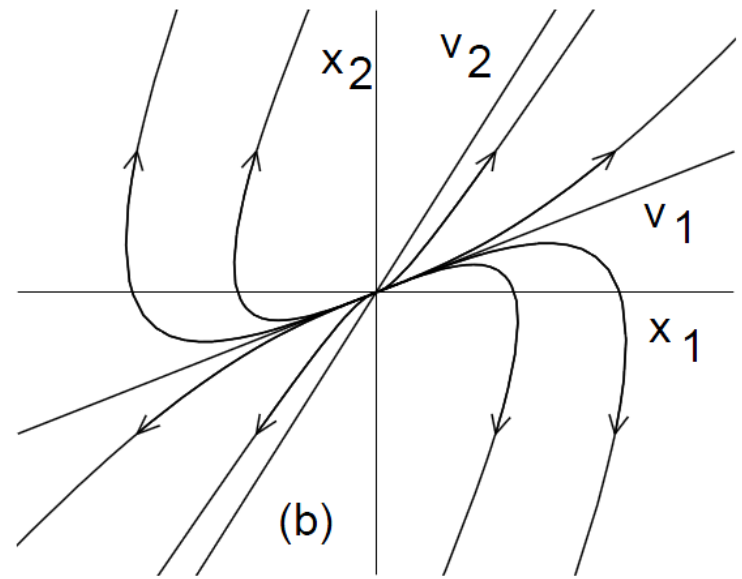
$$\lambda_2 > \lambda_1 > 0$$

Reverse arrowheads

Reverse arrowheads \implies Unstable Node



Stable Node



Unstable Node

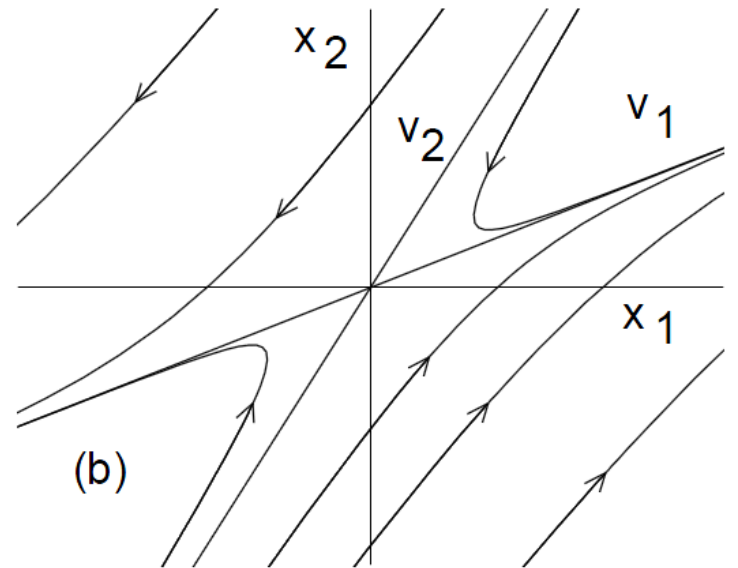
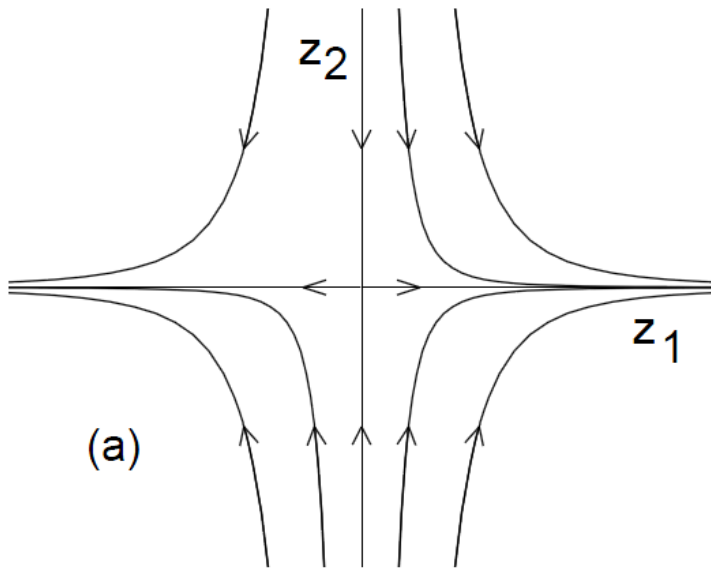
$$\lambda_2 < 0 < \lambda_1$$

$e^{\lambda_1 t} \rightarrow \infty$, while $e^{\lambda_2 t} \rightarrow 0$ as $t \rightarrow \infty$

Call λ_2 the stable eigenvalue (v_2 the stable eigenvector) and λ_1 the unstable eigenvalue (v_1 the unstable eigenvector)

$$z_2 = cz_1^{\lambda_2/\lambda_1}, \quad \lambda_2/\lambda_1 < 0$$

Saddle



Phase Portrait of a Saddle Point

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \quad \dot{z}_2 = \beta z_1 + \alpha z_2$$

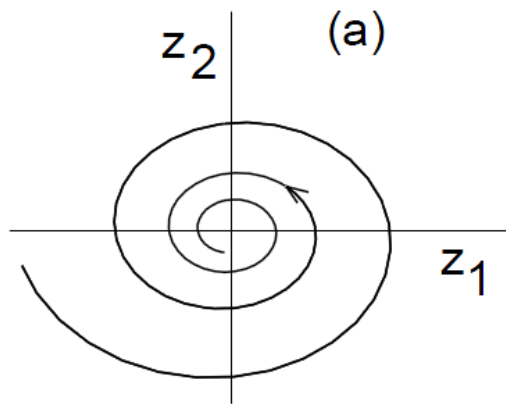
$$r = \sqrt{z_1^2 + z_2^2}, \quad \theta = \tan^{-1} \left(\frac{z_2}{z_1} \right)$$

$$r(t) = r_0 e^{\alpha t} \quad \text{and} \quad \theta(t) = \theta_0 + \beta t$$

$$\alpha < 0 \Rightarrow r(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

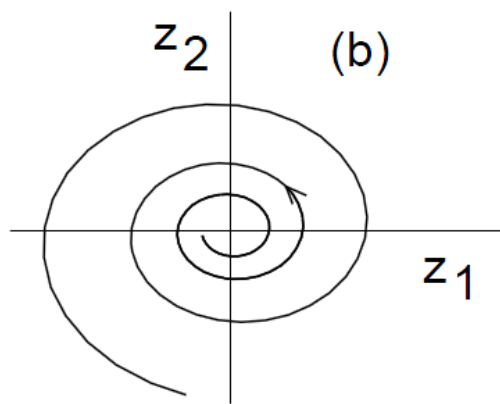
$$\alpha > 0 \Rightarrow r(t) \rightarrow \infty \text{ as } t \rightarrow \infty$$

$$\alpha = 0 \Rightarrow r(t) \equiv r_0 \forall t$$



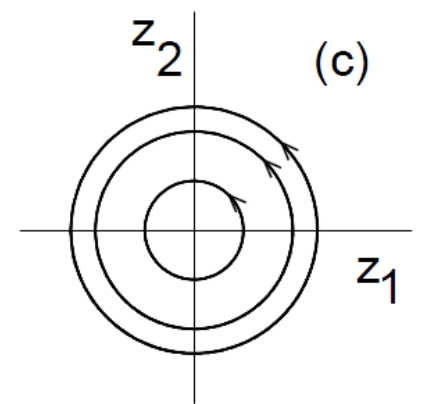
$$\alpha < 0$$

Stable Focus



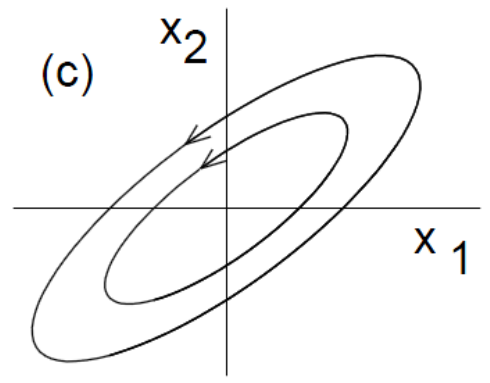
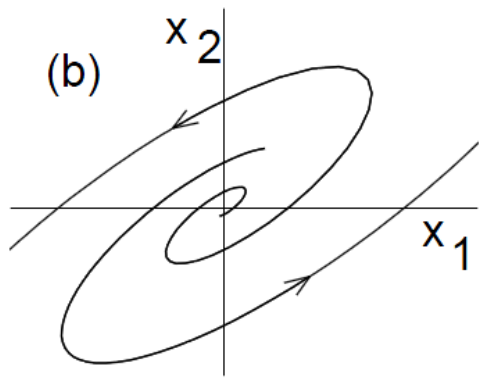
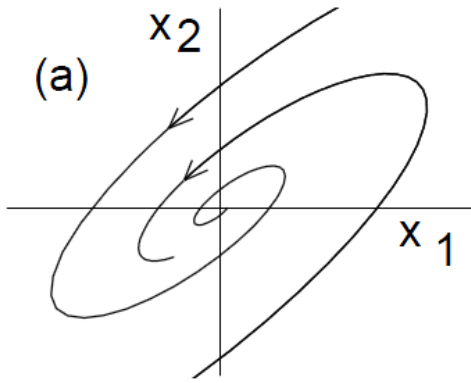
$$\alpha > 0$$

Unstable Focus



$$\alpha = 0$$

Center



Limit Cycles

Oscillation: A system oscillates when it has a **nontrivial periodic solution**

$$x(t + T) = x(t), \quad \forall t \geq 0$$

Linear (Harmonic) Oscillator:

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$$

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \quad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \quad \theta_0 = \tan^{-1} \left[\frac{z_2(0)}{z_1(0)} \right]$$

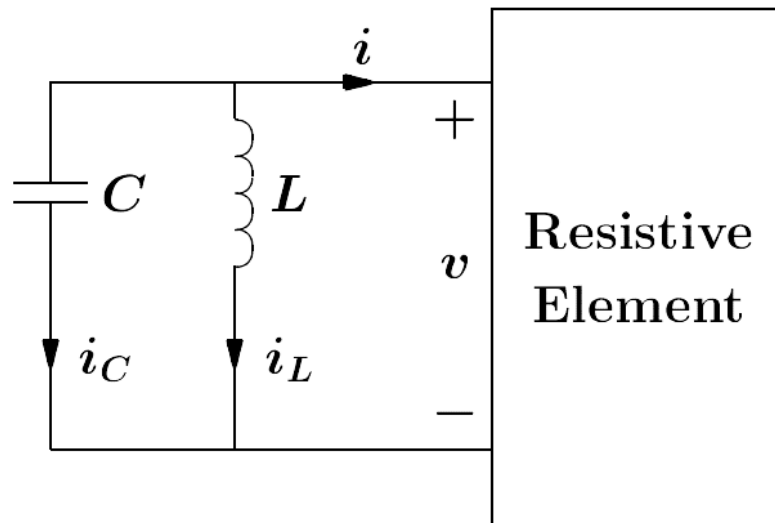
The linear oscillation is not practical because

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation)
- The amplitude of oscillation depends on the initial conditions

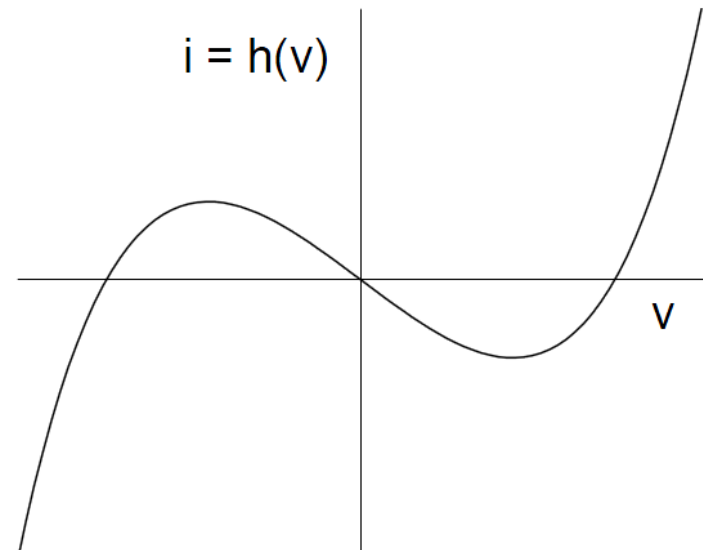
The same problems exist with oscillation of nonlinear systems due to a center equilibrium point (e.g., pendulum without friction)

Limit Cycles:

Example: Negative Resistance Oscillator



(a)



(b)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2\end{aligned}$$

There is a unique equilibrium point at the origin

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}$$

$$\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0$$

$h'(0) < 0 \Rightarrow$ Unstable Focus or Unstable Node

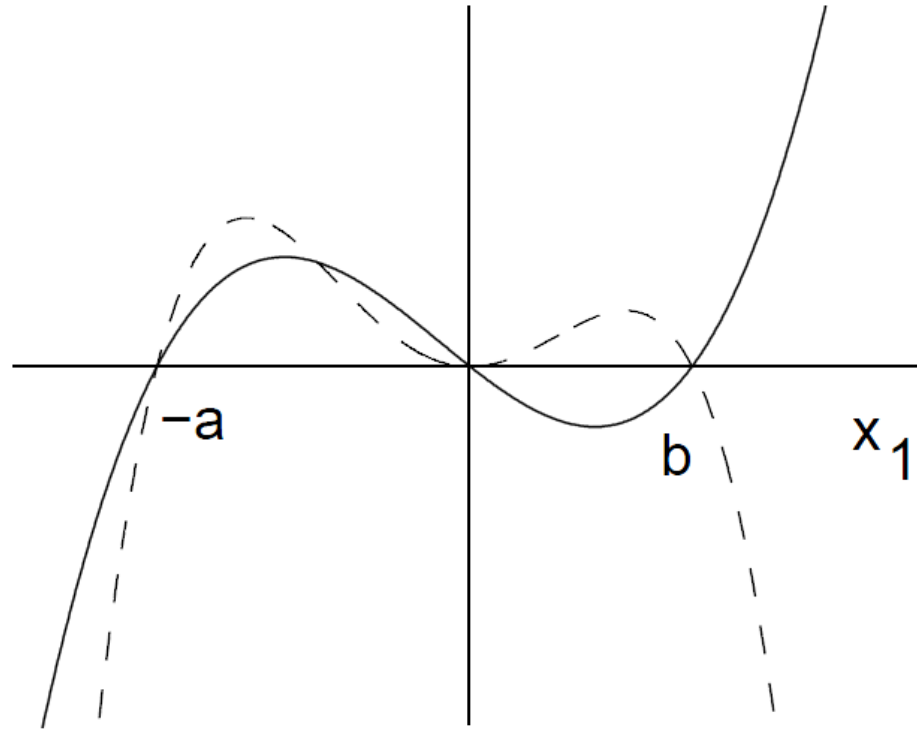
Energy Analysis:

$$E = \frac{1}{2}Cv_C^2 + \frac{1}{2}Li_L^2$$

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon}x_2$$

$$E = \frac{1}{2}C\{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

$$\begin{aligned}\dot{E} &= C\{x_1\dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\ &= C\{x_1x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)x_2 - x_1 - \varepsilon h'(x_1)x_2]\} \\ &= C[x_1x_2 - \varepsilon x_1h(x_1) - x_1x_2] \\ &= -\varepsilon Cx_1h(x_1)\end{aligned}$$

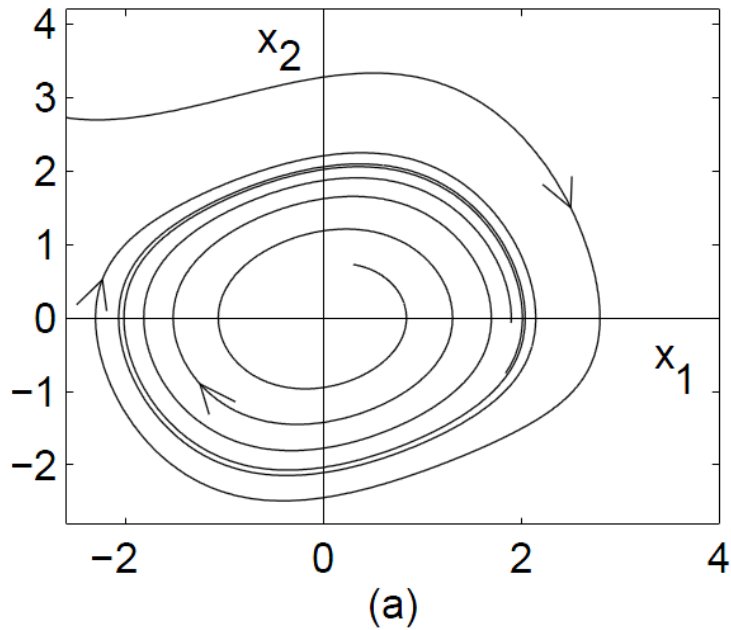


$$\dot{E} = -\varepsilon C x_1 h(x_1)$$

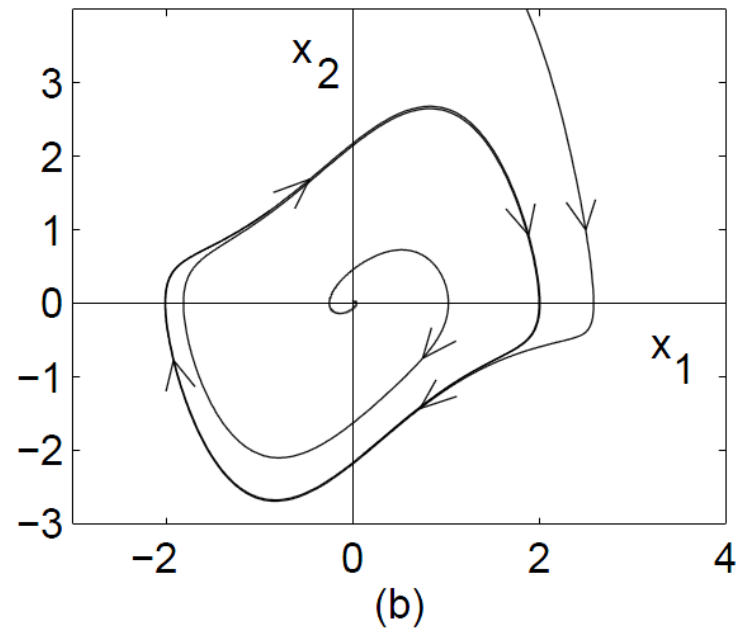
Example: Van der Pol Oscillator

$$\dot{x}_1 = x_2$$

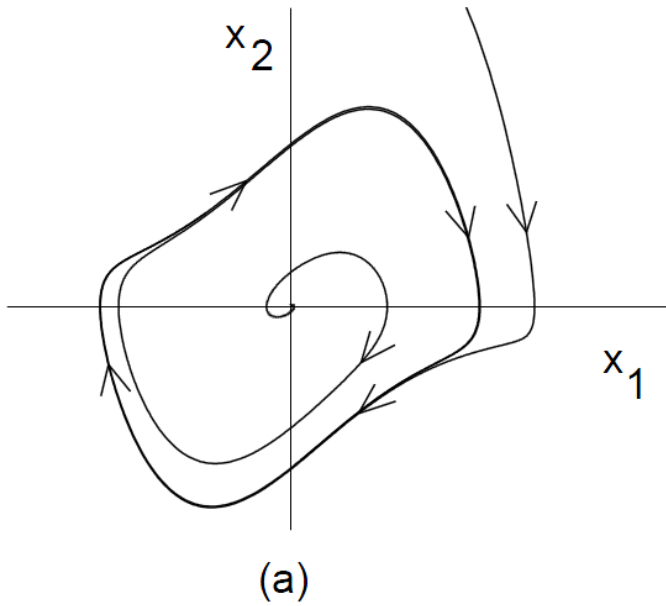
$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2$$



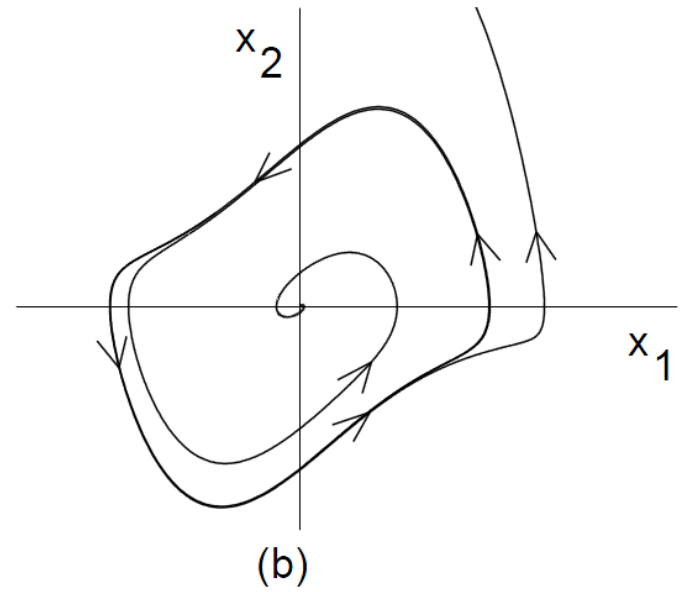
$\varepsilon = 0.2$



$\varepsilon = 1$



Stable Limit Cycle



Unstable Limit Cycle