Chapter 3
Fundamental Properties

Mathematical Review

Euclidean Space

The set of all n-dimensional vectors $x = [x_1, \ldots, x_n]^T$, where $x_1, \ldots, x_n$ are real numbers, defines the n-dimensional Euclidean space denoted by R^n. The one-dimensional Euclidean space consists of all real numbers and is denoted by R. Vectors in R^n can be added by adding their corresponding components. They can be multiplied by a scalar by multiplying each component by the scalar. The inner product of two vectors x and y is $x^T y = \sum_{i=1}^{n} x_i y_i$.

Vector and Matrix Norms

The norm $\|x\|$ of a vector x is a real-valued function with the properties

- $\|x\| \geq 0$ for all $x \in R^n$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|x + y\| \leq \|x\| + \|y\|$, for all $x,\ y \in R^n$.
- $\|\alpha x\| = |\alpha|\ \|x\|$, for all $\alpha \in R$ and $x \in R^n$.

The second property is the triangle inequality. We consider the class of p-norms, defined by

$$\|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}, \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty = \max_i |x_i|$$

The three most commonly used norms are $\|x\|_1$, $\|x\|_\infty$, and the Euclidean norm

$$\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{1/2} = \left(x^T x\right)^{1/2}$$

Matrix Norm

An $m \times n$ matrix A of real elements defines a linear mapping $y = Ax$ from R^n into R^m. The induced p-norm of A is defined by[1]

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p$$

which for $p = 1$, 2, and ∞ is given by

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|, \quad \|A\|_2 = \left[\lambda_{\max}(A^T A)\right]^{1/2}, \quad \text{and} \quad \|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

where $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of $A^T A$. Some useful properties of induced matrix norms for real matrices A and B of dimensions $m \times n$ and $n \times \ell$, respectively, are as follows:

$$\frac{1}{\sqrt{n}} \|A\|_\infty \le \|A\|_2 \le \sqrt{m}\, \|A\|_\infty, \quad \frac{1}{\sqrt{m}} \|A\|_1 \le \|A\|_2 \le \sqrt{n}\, \|A\|_1$$

$$\|A\|_2 \le \sqrt{\|A\|_1 \, \|A\|_\infty}, \quad \|AB\|_p \le \|A\|_p \, \|B\|_p$$

Convergence of Sequences: A sequence of vectors $x_0, x_1, \ldots, x_k, \ldots$ in R^n, denoted by $\{x_k\}$, is said to converge to a limit vector x if

$$\|x_k - x\| \to 0 \text{ as } k \to \infty$$

which is equivalent to saying that, given any $\varepsilon > 0$, there is an integer N such that

$$\|x_k - x\| < \varepsilon, \quad \forall\, k \geq N$$

Sets: A subset $S \subset R^n$ is said to be *open* if, for every vector $x \in S$, one can find an ε-neighborhood of x

$$N(x, \varepsilon) = \{z \in R^n \mid \|z - x\| < \varepsilon\}$$

such that $N(x, \varepsilon) \subset S$. A set S is *closed* if and only if its complement in R^n is open. Equivalently, S is closed if and only if every convergent sequence $\{x_k\}$ with elements in S converges to a point in S. A set S is *bounded* if there is $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. A set S is *compact* if it is closed and bounded. A point p is a *boundary point* of a set S if every neighborhood of p contains at least one point of S and one point not belonging to S. The set of all boundary points of S, denoted by ∂S, is called the boundary of S. A closed set contains all its boundary points. An open set contains none of its boundary points. The *interior* of a set S is $S - \partial S$. An open set is equal to its interior. The *closure* of a set S, denoted by $\bar{S}$, is the union of S and its boundary. A closed set is equal to its closure. An open set S is connected if every pair of points in S can be joined by an arc lying in S. A set S is called a *region* if it is the union of an open connected set with some, none, or all of its boundary points. If none of the boundary points are included, the region is called an open region or *domain*. A set S is *convex* if, for every $x, y \in S$ and every real number θ, $0 < \theta < 1$, the point $\theta x + (1 - \theta)y \in S$. If $x \in X \subset R^n$ and $y \in Y \subset R^m$, we say that (x, y) belongs to the product set $X \times Y \subset R^n \times R^m$.

Continuous Functions: A function f mapping a set S_1 into a set S_2 is denoted by $f : S_1 \to S_2$. A function $f : R^n \to R^m$ is said to be *continuous* at a point x if $f(x_k) \to f(x)$ whenever $x_k \to x$. Equivalently, f is continuous at x if, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|x - y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

The symbol "$\Rightarrow$" reads "implies." A function f is continuous on a set S if it is continuous at every point of S, and it is *uniformly continuous* on S if, given $\varepsilon > 0$ there is $\delta > 0$ (dependent only on ε) such that the inequality holds for all $x,\ y \in S$. Note that uniform continuity is defined on a set, while continuity is defined at a point. For uniform continuity, the same constant δ works for all points in the set.

Differentiable functions: A function $f : R \to R$ is said to be *differentiable* at x if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. The limit $f'(x)$ is called the derivative of f at x. A function $f : R^n \to R^m$ is said to be *continuously differentiable* at a point x_0 if the partial derivatives $\partial f_i / \partial x_j$ exist and are continuous at x_0 for $1 \leq i \leq m$, $1 \leq j \leq n$. A function f is continuously differentiable on a set S if it is continuously differentiable at every point of S. For a continuously differentiable function $f : R^n \to R$, the row vector $\partial f / \partial x$ is defined by

$$\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right]$$

The *gradient vector*, denoted by $\nabla f(x)$, is

$$\nabla f(x) = \left[\frac{\partial f}{\partial x} \right]^T$$

For a continuously differentiable function $f : R^n \to R^m$, the *Jacobian matrix* $[\partial f / \partial x]$ is an $m \times n$ matrix whose element in the ith row and jth column is $\partial f_i / \partial x_j$.

Existence and Uniqueness of Solutions

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x over the domain of interest

$f(t, x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities

$f(t, x)$ is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0, r) = \{x \in R^n \mid \|x - x_0\| < r\}$ where $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\|, \ \ L > 0$$

A function $f(t, x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$

When $n = 1$ and f depends only on x

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

On a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L

Any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point

A discontinuous function is not locally Lipschitz at the points of discontinuity

The function $f(x) = x^{1/3}$ is not locally Lipschitz at $x = 0$ since

$$f'(x) = (1/3)x^{-2/3} \rightarrow \infty \text{ a } x \rightarrow 0$$

On the other hand, if $f'(x)$ is continuous at a point x_0 then $f(x)$ is locally Lipschitz at the same point because continuity of $f'(x)$ ensures that $|f'(x)|$ is bounded by a constant k in a neighborhood of x_0 ; which implies that $f(x)$ satisfies the Lipschitz condition $L = k$

More generally, if for $t \in J \subset R$ and x in a domain $D \subset R^n$, $f(t, x)$ and its partial derivatives $\partial f_i / \partial x_j$ are continuous, then $f(t, x)$ is locally Lipschitz in x on D

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$

Without the local Lipschitz condition, we cannot ensure uniqueness of the solution. For example, $\dot{x} = x^{1/3}$ has $x(t) = (2t/3)^{3/2}$ and $x(t) \equiv 0$ as two different solutions when the initial state is $x(0) = 0$

The lemma is a local result because it guarantees existence and uniqueness of the solution over an interval $[t_0, t_0 + \delta]$, but this interval might not include a given interval $[t_0, t_1]$. Indeed the solution may cease to exist after some time

Example:

$$\dot{x} = -x^2$$

$f(x) = -x^2$ is locally Lipschitz for all x

$$x(0) = -1 \ \Rightarrow \ x(t) = \frac{1}{(t-1)}$$

$$x(t) \rightarrow -\infty \ \text{ as } \ t \rightarrow 1$$

the solution has a *finite escape time* at $t = 1$

In general, if $f(t, x)$ is locally Lipschitz over a domain D and the solution of $\dot{x} = f(t, x)$ has a finite escape time t_e, then the solution $x(t)$ must leave every compact (closed and bounded) subset of D as $t \rightarrow t_e$

Global Existence and Uniqueness

A function $f(t, x)$ is globally Lipschitz in x if

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\|$$

for all $x, y \in R^n$ with the same Lipschitz constant L

If $f(t, x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous for all $x \in R^n$, then $f(t, x)$ is globally Lipschitz in x if and only if the partial derivatives $\partial f_i/\partial x_j$ are globally bounded, uniformly in t

$f(x) = -x^2$ is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded

Lemma: Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$

The global Lipschitz condition is satisfied for linear systems of the form

$$\dot{x} = A(t)x + g(t)$$

but it is a restrictive condition for general nonlinear systems

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, and suppose that every solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

with $x_0 \in W$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$

Example:

$$\dot{x} = -x^3 = f(x)$$

$f(x)$ is locally Lipschitz on R, but not globally Lipschitz because $f'(x) = -3x^2$ is not globally bounded

If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive

Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in R \mid |x| \leq |a|\}$

Thus, the equation has a unique solution for all $t \geq 0$