Chapter 4
Lyapunov Stability

$$\dot{x} = f(x)$$

f is locally Lipschitz over a domain $D \subset R^n$

Suppose $\bar{x} \in D$ is an equilibrium point; that is, $f(\bar{x}) = 0$

Characterize and study the stability of $\bar{x}$

For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of R^n; that is, $\bar{x} = 0$. No loss of generality

$$y = x - \bar{x}$$

$$\dot{y} = \dot{x} = f(x) = f(y + \bar{x}) \stackrel{\text{def}}{=} g(y), \quad \text{where } g(0) = 0$$

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is

- stable if for each $\varepsilon > 0$ there is $\delta > 0$ (dependent on ε) such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \ \ \forall\, t \geq 0$$

- unstable if it is not stable

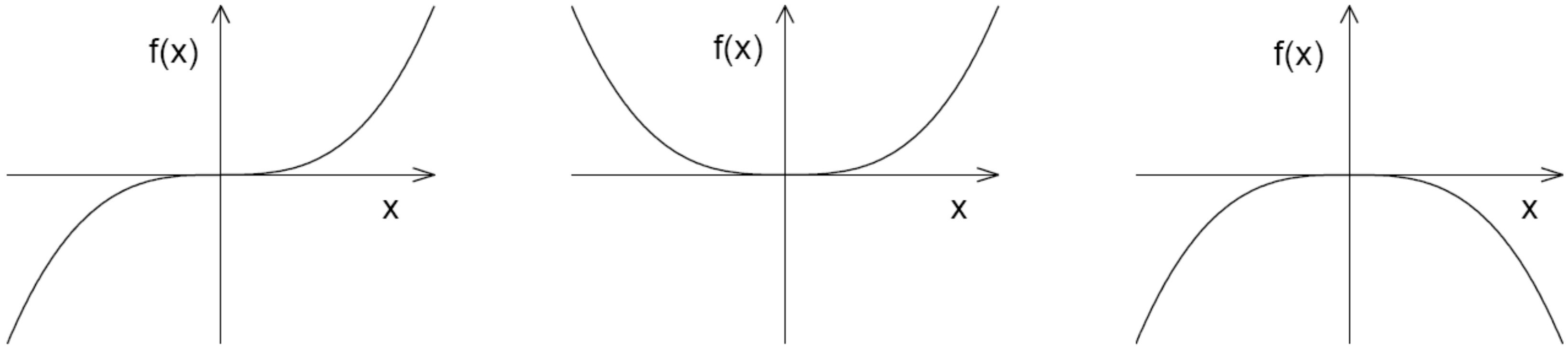
- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0$$

First-Order Systems $(n = 1)$

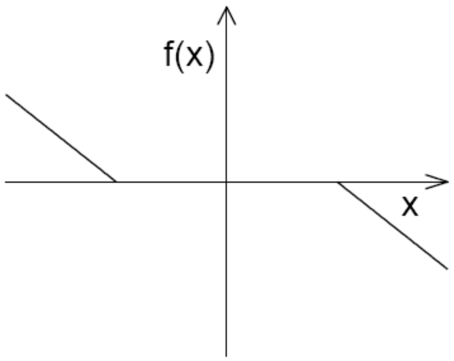
The behavior of $x(t)$ in the neighborhood of the origin can be determined by examining the sign of $f(x)$

The ε–δ requirement for stability is violated if $xf(x) > 0$ on either side of the origin

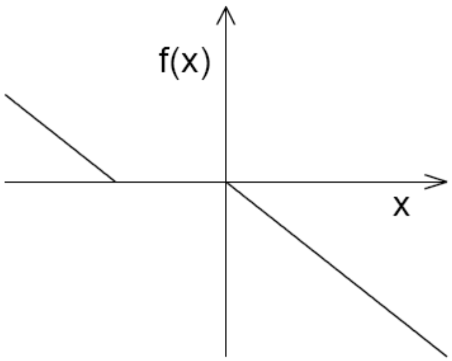


Unstable Unstable Unstable

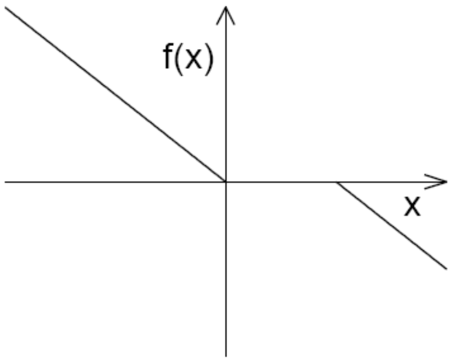
The origin is stable if and only if $xf(x) \leq 0$ in some neighborhood of the origin



Stable

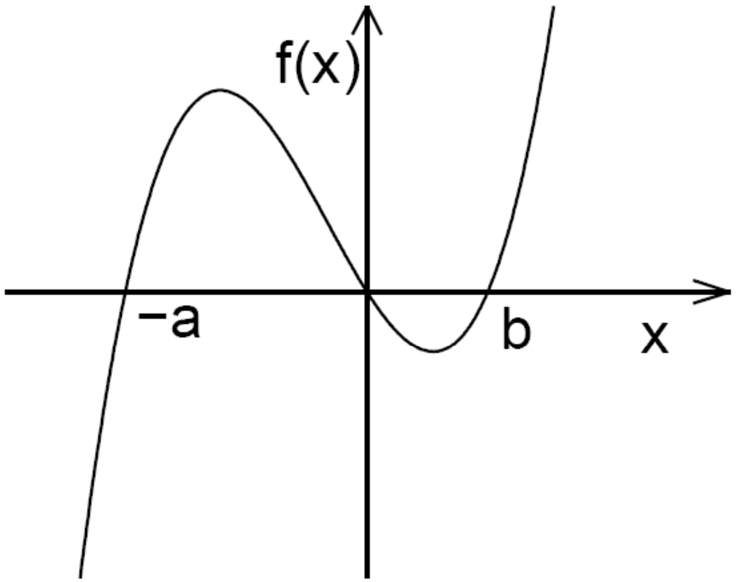


Stable

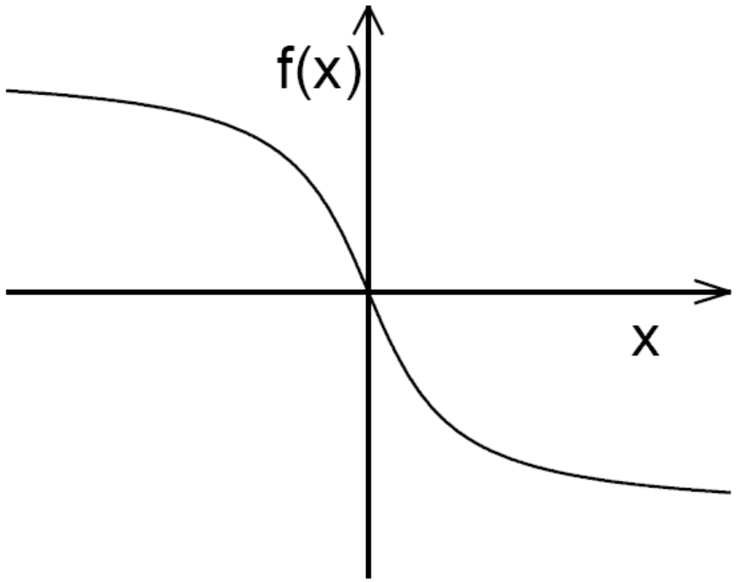


Stable

The origin is asymptotically stable if and only if $xf(x) < 0$ in some neighborhood of the origin



(a)



(b)

Asymptotically Stable

Globally Asymptotically Stable

Definition: Let the origin be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where f is a locally Lipschitz function defined over a domain $D \subset R^n$ ($0 \in D$)

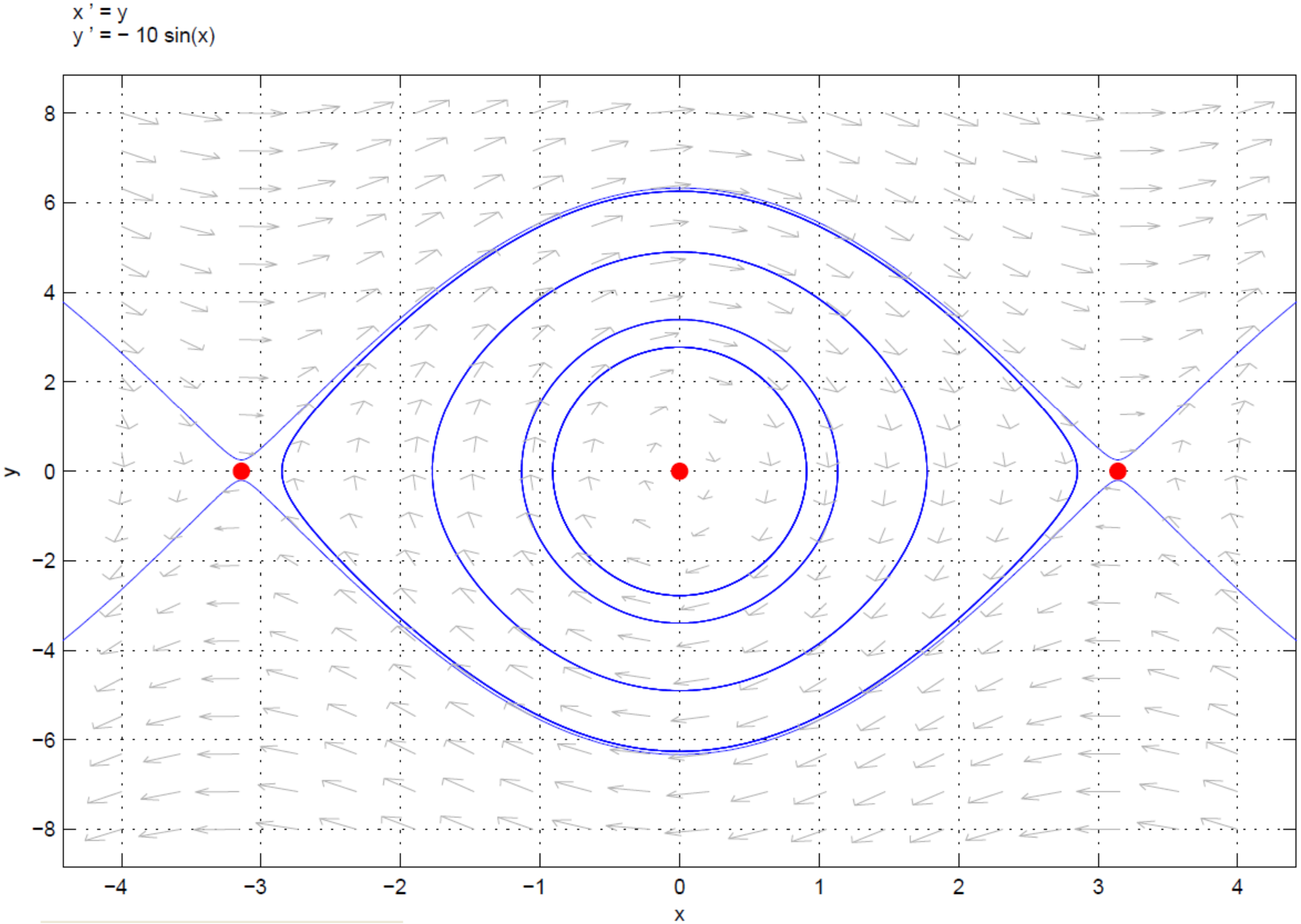
- The region of attraction (also called region of asymptotic stability, domain of attraction, or basin) is the set of all points x_0 in D such that the solution of

$$\dot{x} = f(x), \quad x(0) = x_0$$

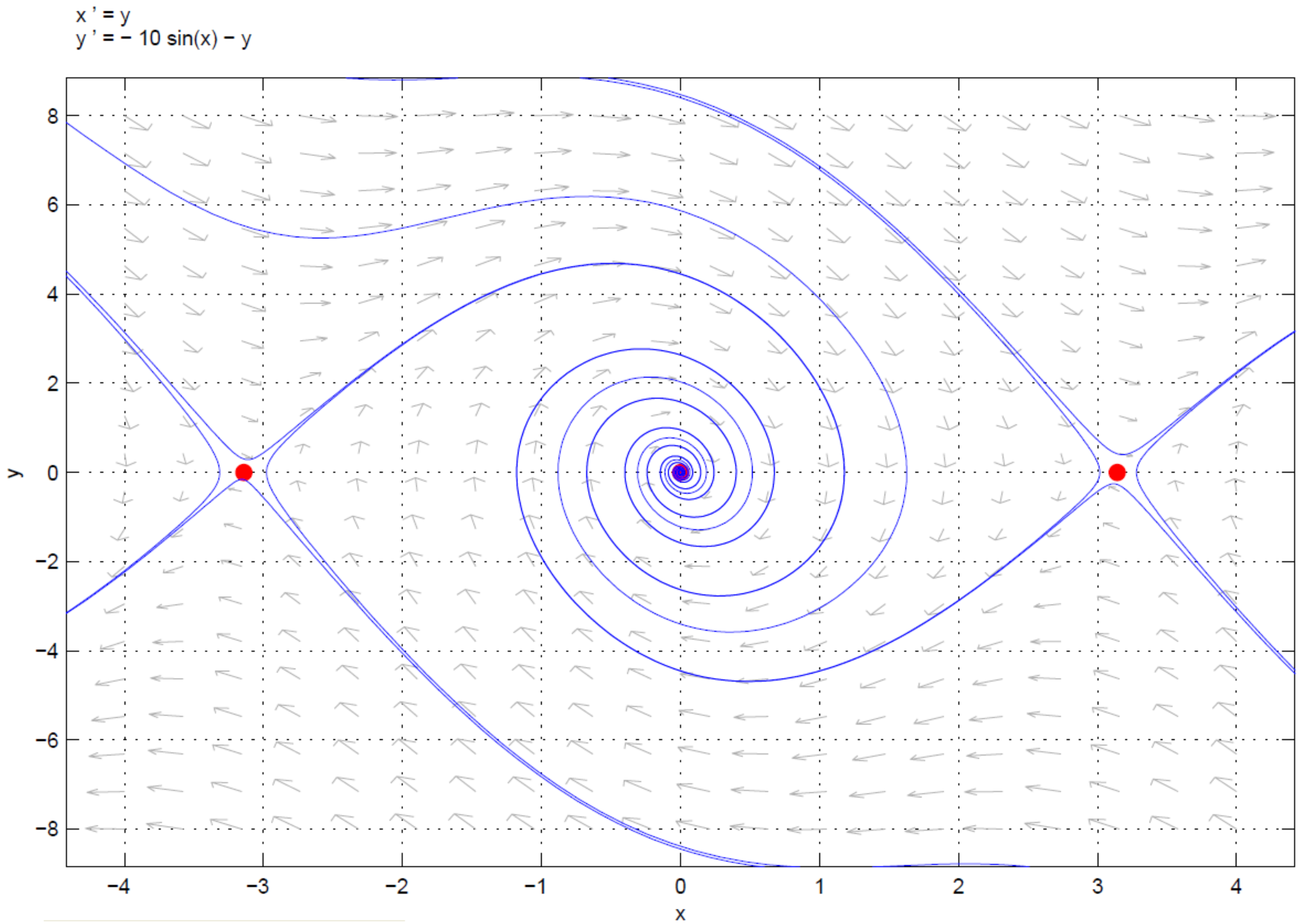
  is defined for all $t \geq 0$ and converges to the origin as t tends to infinity

- The origin is said to be globally asymptotically stable if the region of attraction is the whole space R^n

Example: Pendulum Without Friction



Example: Pendulum With Friction



Linear Time-Invariant Systems

$$\dot{x} = Ax$$

$$x(t) = \exp(At)x(0)$$

$$P^{-1}AP = J = \text{block diag}[J_1, J_2, \ldots, J_r]$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda_i & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \ldots & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}_{m \times m}$$

$$\exp(\boldsymbol{A}t) = \boldsymbol{P}\exp(\boldsymbol{J}t)\boldsymbol{P}^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\boldsymbol{\lambda}_i t)\boldsymbol{R}_{ik}$$

m_i is the order of the Jordan block J_i

$\mathrm{Re}[\boldsymbol{\lambda}_i] < 0 \ \forall\, i \ \Leftrightarrow$ Asymptotically Stable

$\mathrm{Re}[\boldsymbol{\lambda}_i] > 0$ for some $i \ \Rightarrow$ Unstable

$\mathrm{Re}[\boldsymbol{\lambda}_i] \leq 0 \ \forall\, i \ \&\ m_i > 1$ for $\mathrm{Re}[\boldsymbol{\lambda}_i] = 0 \ \Rightarrow$ Unstable

$\mathrm{Re}[\boldsymbol{\lambda}_i] \leq 0 \ \forall\, i \ \&\ m_i = 1$ for $\mathrm{Re}[\boldsymbol{\lambda}_i] = 0 \ \Rightarrow$ Stable

If an $n \times n$ matrix A has a repeated eigenvalue λ_i of algebraic multiplicity q_i, then the Jordan blocks of λ_i have order one if and only if $\mathrm{rank}(A - \lambda_i I) = n - q_i$

Theorem: The equilibrium point $x = 0$ of $\dot{x} = Ax$ is stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] \le 0$ and for every eigenvalue with $\mathrm{Re}[\lambda_i] = 0$ and algebraic multiplicity $q_i \ge 2$, $\mathrm{rank}(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point $x = 0$ is globally asymptotically stable if and only if all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$

When all eigenvalues of A satisfy $\mathrm{Re}[\lambda_i] < 0$, A is called a *Hurwitz matrix*

When the origin of a linear system is asymptotically stable, its solution satisfies the inequality

$$\|x(t)\| \le k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \ge 0$$

$k \ge 1,\ \lambda > 0$

Exponential Stability

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(x)$ is said to be exponentially stable if

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, t \geq 0$$

$k \geq 1,\ \lambda > 0$, for all $\|x(0)\| < c$

It is said to be globally exponentially stable if the inequality is satisfied for any initial state $x(0)$

Exponential Stability $\Rightarrow$ Asymptotic Stability

Example

$$\dot{x} = -x^3$$

The origin is asymptotically stable

$$x(t) = \frac{x(0)}{\sqrt{1 + 2tx^2(0)}}$$

$x(t)$ does not satisfy $|x(t)| \le ke^{-\lambda t}|x(0)|$ because

$$x(t)| \le ke^{-\lambda t}|x(0)| \;\Rightarrow\; \frac{e^{2\lambda t}}{1 + 2tx^2(0)} \le k^2$$

Impossible because $\lim_{t\to\infty} \frac{e^{2\lambda t}}{1 + 2tx^2(0)} = \infty$

Linearization

$$\dot{x} = f(x), \quad f(0) = 0$$

f is continuously differentiable over $D = \{\|x\| < r\}$

$$J(x) = \frac{\partial f}{\partial x}(x)$$

$$h(\sigma) = f(\sigma x) \text{ for } 0 \le \sigma \le 1$$

$$h'(\sigma) = J(\sigma x)x$$

$$h(1) - h(0) = \int_0^1 h'(\sigma)\, d\sigma, \quad h(0) = f(0) = 0$$

$$f(x) = \int_0^1 J(\sigma x)\, d\sigma\; x$$

$$f(x) = \int_0^1 J(\sigma x)\, d\sigma\ x$$

Set $A = J(0)$ and add and subtract Ax

$f(x) = [A + G(x)]x$, where $G(x) = \int_0^1 [J(\sigma x) - J(0)]\, d\sigma$

$$G(x) \to 0 \ \text{ as } \ x \to 0$$

This suggests that in a small neighborhood of the origin we can approximate the nonlinear system $\dot{x} = f(x)$ by its linearization about the origin $\dot{x} = Ax$

Theorem:

- The origin is exponentially stable if and only if $\mathrm{Re}[\boldsymbol{\lambda}_i] < 0$ for all eigenvalues of $\boldsymbol{A}$

- The origin is unstable if $\mathrm{Re}[\boldsymbol{\lambda}_i] > 0$ for some i

Linearization fails when $\mathrm{Re}[\boldsymbol{\lambda}_i] \leq 0$ for all i, with $\mathrm{Re}[\boldsymbol{\lambda}_i] = 0$ for some i

Example

$$\dot{x} = ax^3$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left.3ax^2\right|_{x=0} = 0$$

Stable if $a = 0$; Asymp stable if $a < 0$; Unstable if $a > 0$

When $a < 0$, the origin is not exponentially stable

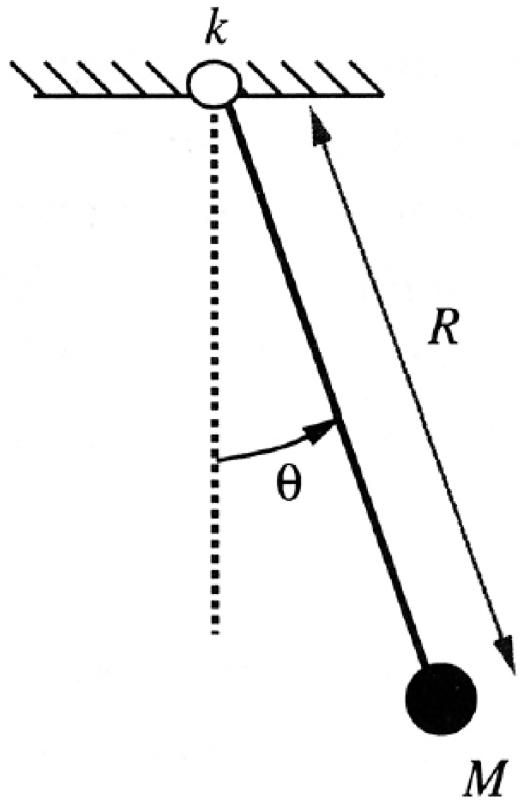
Example: The Pendulum

$$MR^2 \ddot{\theta} + k \dot{\theta} + MgR \sin \theta = 0$$

$x_1 = \theta$, $x_2 = \dot{\theta}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{MR^2} x_2 - \frac{g}{R} \sin x_1$$



$x_2 = 0$, $\sin x_1 = 0$

$(0\ [2\pi], 0)$ and $(\pi\ [2\pi], 0)$.

Example : The Pendulum(1)

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{b}{MR^2} x_2 - \frac{g}{R} \sin x_1$$

$$\frac{\partial f_1}{\partial x_1} = 0, \frac{\partial f_1}{\partial x_2} = 1, \frac{\partial f_2}{\partial x_1} = -\frac{g}{R} \cos x_1, \frac{\partial f_2}{\partial x_2} = -\frac{b}{MR^2}$$

$$x_1^* = 0, x_2^* = 0, A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{R} & -\frac{b}{MR^2} \end{bmatrix}$$

$$s^2 + \frac{b}{MR^2} s + \frac{g}{R} = 0$$

Example: The Pendulum(2)

$$x_1^* = \pi, x_2^* = 0, A = \begin{bmatrix} 0 & 1 \\ \dfrac{g}{R} & -\dfrac{b}{MR^2} \end{bmatrix}$$

$$s^2 + \frac{b}{MR^2}s - \frac{g}{R} = 0$$

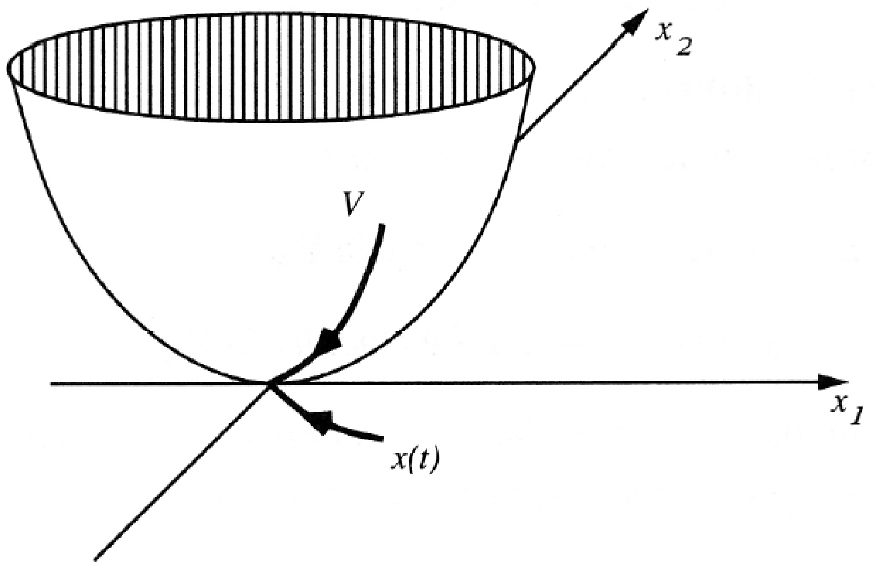
$$z_1 = x_1 - x_1^* = x_1 - \pi, z_2 = x_2$$

$$\dot{z}_1 = z_2$$

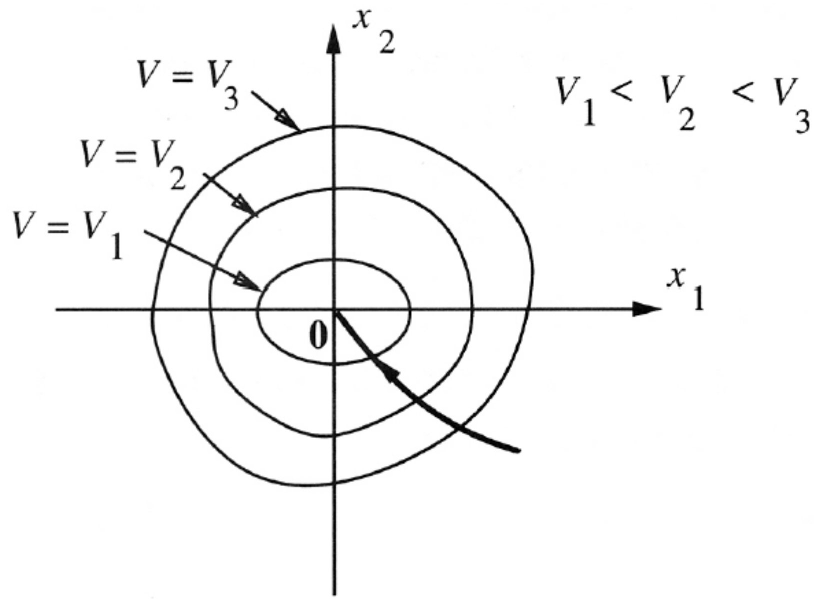
$$\dot{z}_2 = -\frac{b}{MR^2}z_2 - \frac{g}{R}\sin(z_1 + \pi)$$

Let $V(x)$ be a continuously differentiable function defined in a domain $D \subset R^n$; $0 \in D$. The derivative of V along the trajectories of $\dot{x} = f(x)$ is

$$
\begin{aligned}
\dot{V}(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \\
&= \left[\begin{array}{cccc} \frac{\partial V}{\partial x_1}, & \frac{\partial V}{\partial x_2}, & \ldots, & \frac{\partial V}{\partial x_n} \end{array} \right] \left[\begin{array}{c} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{array} \right] \\
&= \frac{\partial V}{\partial x} f(x)
\end{aligned}
$$



x 2
V
x 1
x(t)



x 2
V = V 3
V 1 < V 2 < V 3
V = V 2
V = V 1
x 1
0

If $\phi(t;x)$ is the solution of $\dot{x} = f(x)$ that starts at initial state x at time $t = 0$, then

$$\dot{V}(x) = \frac{d}{dt} V(\phi(t;x)) \bigg|_{t=0}$$

If $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$

If $\dot{V}(x)$ is positive, V will increase along the solution of $\dot{x} = f(x)$

Lyapunov's Theorem:

- If there is $V(x)$ such that

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall\, x \in D/\{0\}$$

$$\dot{V}(x) \leq 0, \quad \forall\, x \in D$$

  then the origin is a stable

- Moreover, if

$$\dot{V}(x) < 0, \quad \forall\, x \in D/\{0\}$$

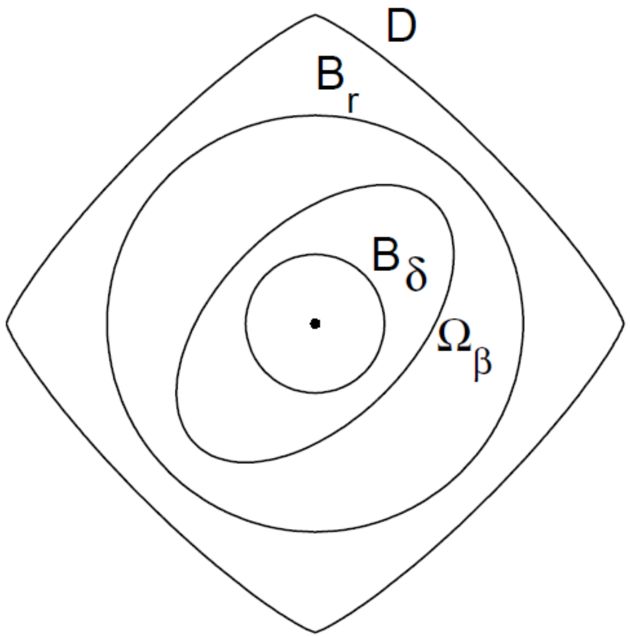
  then the origin is asymptotically stable

- Furthermore, if $V(x) > 0$, $\forall\ x \neq 0$,

$$\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty$$

and $\dot{V}(x) < 0$, $\forall\ x \neq 0$, then the origin is globally asymptotically stable

Proof:



$$0 < r \leq \varepsilon, \ \ B_r = \{\|x\| \leq r\}$$

$$\alpha = \min_{\|x\|=r} V(x) > 0$$

$$0 < \beta < \alpha$$

$$\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$$

$$\|x\| \leq \delta \ \Rightarrow \ V(x) < \beta$$

Solutions starting in Ω_β stay in Ω_β because $\dot{V}(x) \le 0$ in Ω_β

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \le \varepsilon, \quad \forall\, t \ge 0$$

$\Rightarrow$ The origin is stable

Now suppose $\dot{V}(x) < 0 \; \forall\, x \in D/\{0\}$. $V(x(t)$ is monotonically decreasing and $V(x(t)) \ge 0$

$$\lim_{t\to\infty} V(x(t)) = c \ge 0$$

$$\lim_{t\to\infty} V(x(t)) = c \ge 0 \quad \text{Show that } c = 0$$

Suppose $c > 0$. By continuity of $V(x)$, there is $d > 0$ such that $B_d \subset \Omega_c$. Then, $x(t)$ lies outside B_d for all $t \ge 0$

$$\gamma = -\max_{d \le \|x\| \le r} \dot{V}(x)$$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \le V(x(0)) - \gamma t$$

This inequality contradicts the assumption $c > 0$

$\Rightarrow$ The origin is asymptotically stable

The condition $\|x\| \to \infty \ \Rightarrow \ V(x) \to \infty$ implies that the set $\Omega_c = \{x \in R^n \mid V(x) \le c\}$ is compact for every $c > 0$. This is so because for any $c > 0$, there is $r > 0$ such that $V(x) > c$ whenever $\|x\| > r$. Thus, $\Omega_c \subset B_r$. All solutions starting Ω_c will converge to the origin. For any point $p \in R^n$, choosing $c = V(p)$ ensures that $p \in \Omega_c$

$\Rightarrow$ The origin is globally asymptotically stable

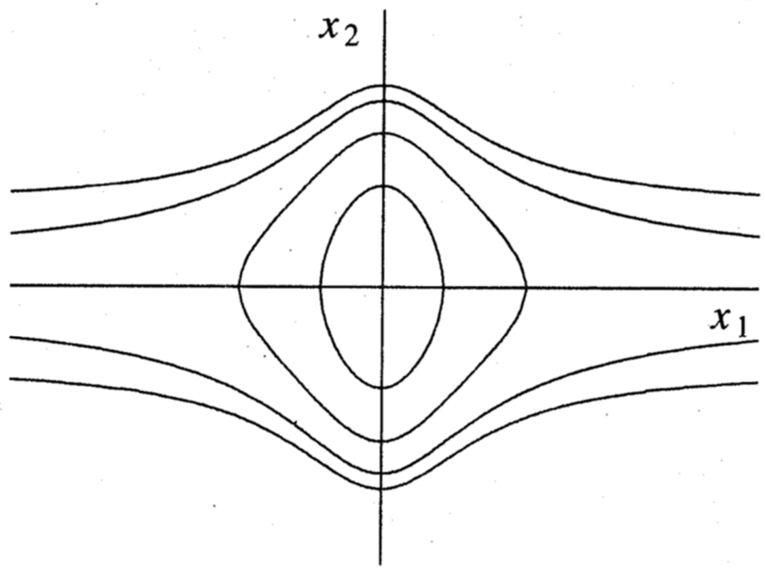
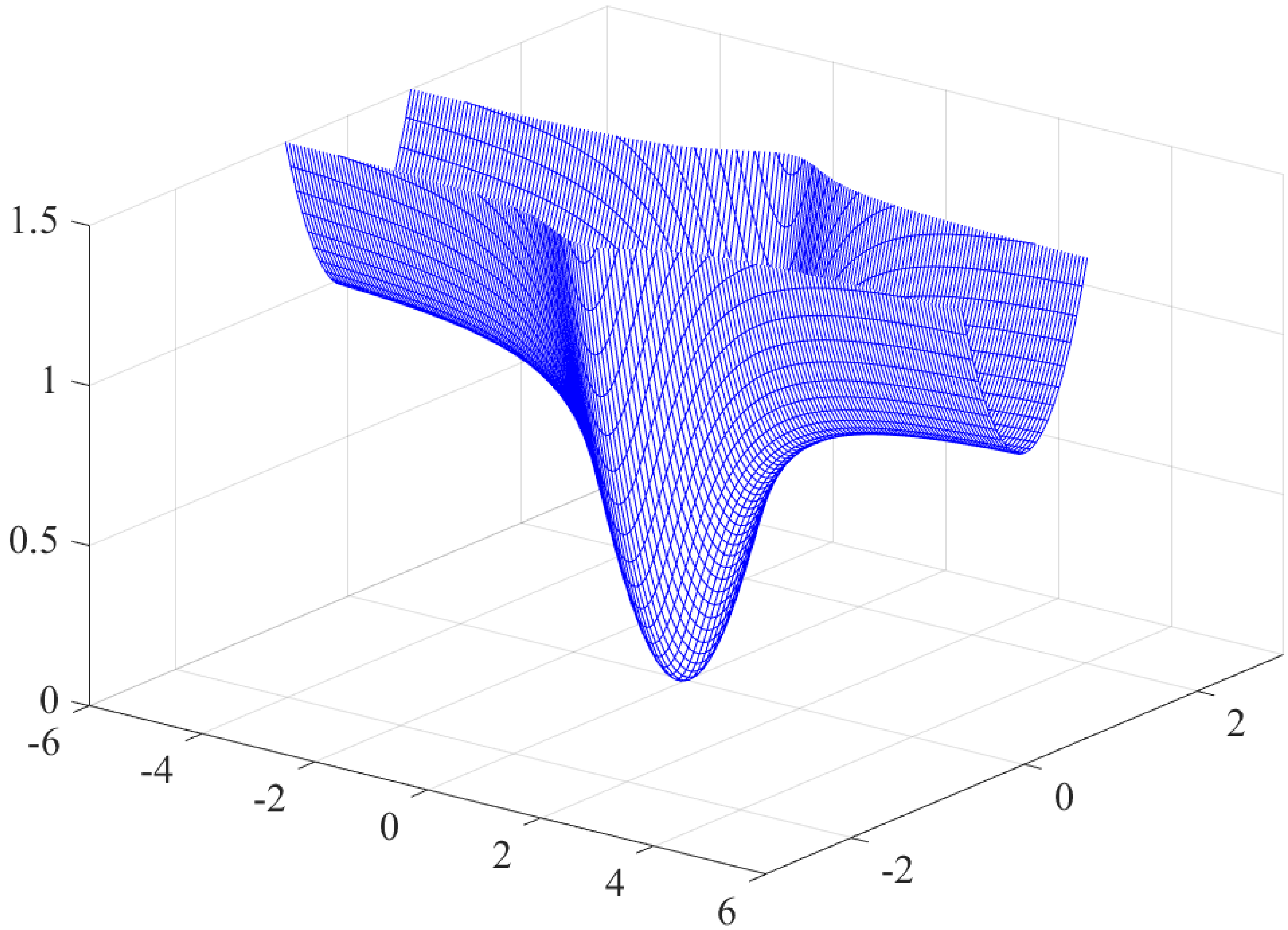


Figure 4.4: Lyapunov surfaces for $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

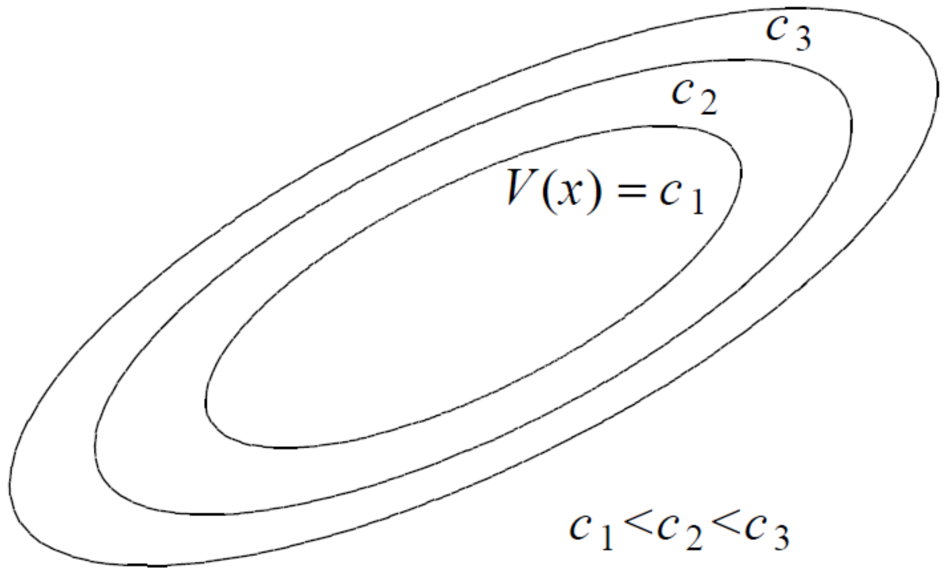

1.5
1
0.5
0
-6
-4
-2
0
2
4
6
-2
0
2

Terminology

$V(0) = 0,\ V(x) \geq 0$ for $x \neq 0$	Positive semidefinite
$V(0) = 0,\ V(x) > 0$ for $x \neq 0$	Positive definite
$V(0) = 0,\ V(x) \leq 0$ for $x \neq 0$	Negative semidefinite
$V(0) = 0,\ V(x) < 0$ for $x \neq 0$	Negative definite
$\lVert x \rVert \to \infty \ \Rightarrow\ V(x) \to \infty$	Radially unbounded

Lyapunov' Theorem: The origin is stable if there is a continuously differentiable positive definite function $V(x)$ so that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite. It is globally asymptotically stable if the conditions for asymptotic stability hold globally and $V(x)$ is radially unbounded

A continuously differentiable function $V(x)$ satisfying the conditions for stability is called a *Lyapunov function*. The surface $V(x) = c$, for some $c > 0$, is called a *Lyapunov surface* or a *level surface*



Quadratic Forms

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \quad P = P^T$$

$$\lambda_{\min}(P) \|x\|^2 \le x^T P x \le \lambda_{\max}(P) \|x\|^2$$

$P \ge 0$ (Positive semidefinite) if and only if $\lambda_i(P) \ge 0 \ \forall i$

$P > 0$ (Positive definite) if and only if $\lambda_i(P) > 0 \ \forall i$

$V(x)$ is positive definite if and only if P is positive definite
$V(x)$ is positive semidefinite if and only if P is positive semidefinite
$P > 0$ if and only if all the leading principal minors of P are positive

Linear Systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x, \qquad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x \stackrel{\text{def}}{=} -x^T Q x$$

If $Q > 0$, then A is Hurwitz

Or choose $Q > 0$ and solve the Lyapunov equation

$$PA + A^T P = -Q$$

If $P > 0$, then A is Hurwitz

Matlab: $P = \text{lyap}(A', Q)$

Theorem A matrix A is Hurwitz if and only if for any $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

Idea of the proof: Sufficiency follows from Lyapunov's theorem. Necessity is shown by verifying that

$$P = \int_0^\infty \exp(A^T t) Q \exp(At)\, dt$$

is positive definite and satisfies the Lyapunov equation

Linearization

$$\dot{x} = f(x) = [A + G(x)]x$$

$$G(x) \to 0 \ \text{ as } \ x \to 0$$

Suppose A is Hurwitz. Choose $Q = Q^T > 0$ and solve the Lyapunov equation $PA + A^T P = -Q$ for P. Use $V(x) = x^T P x$ as a Lyapunov function candidate for $\dot{x} = f(x)$

$$\begin{aligned}
\dot{V}(x) &= x^T P f(x) + f^T(x) P x \\
&= x^T P[A + G(x)]x + x^T[A^T + G^T(x)]Px \\
&= x^T(PA + A^T P)x + 2x^T PG(x)x \\
&= -x^T Q x + 2x^T PG(x)x
\end{aligned}$$

$$\dot{V}(x) \le -x^T Q x + 2\|P\|\ \|G(x)\|\ \|x\|^2$$

For any $\gamma > 0$, there exists $r > 0$ such that

$$\|G(x)\| < \gamma, \ \ \forall\ \|x\| < r$$

$$x^T Q x \ge \lambda_{\min}(Q)\|x\|^2 \ \Leftrightarrow\ -x^T Q x \le -\lambda_{\min}(Q)\|x\|^2$$

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma\|P\|]\|x\|^2, \ \ \forall\ \|x\| < r$$

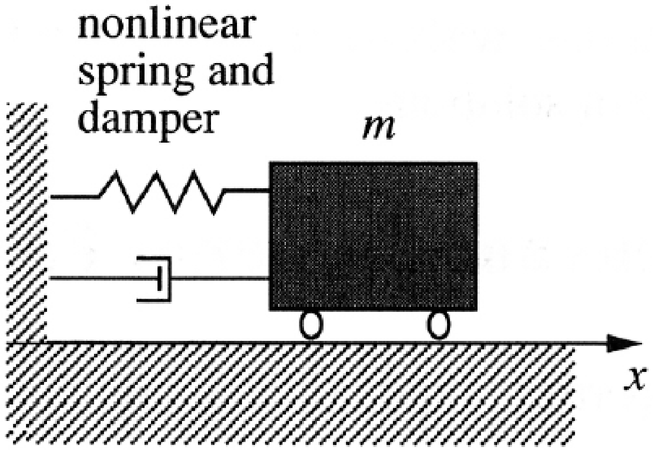
Choose

$$\gamma < \frac{\lambda_{\min}(Q)}{2\|P\|}$$

$V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = f(x)$

Example

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_o x + k_1 x^3 = 0$$



$$V(\mathbf{x}) = \frac{1}{2} m\dot{x}^2 + \int_o^x (k_o x + k_1 x^3)\, dx = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} k_o x^2 + \frac{1}{4} k_1 x^4$$

- zero energy corresponds to the equilibrium point ($\mathbf{x} = \mathbf{0}, \dot{\mathbf{x}} = \mathbf{0}$)
- asymptotic stability implies the convergence of mechanical energy to zero
- instability is related to the growth of mechanical energy

$$\dot{V}(\mathbf{x}) = m\dot{x}\ddot{x} + (k_o x + k_1 x^3)\,\dot{x} = \dot{x}\,(-b\dot{x}|\dot{x}|) = -b|\dot{x}|^3$$

Example:

$$\dot{x} = -g(x)$$

$$g(0) = 0; \;\; xg(x) > 0, \;\; \forall\, x \neq 0 \;\; \text{and} \;\; x \in (-a, a)$$

$$V(x) = \int_0^x g(y)\, dy$$

$$\dot{V}(x) = \frac{\partial V}{\partial x}[-g(x)] = -g^2(x) < 0, \; \forall\, x \in (-a, a), \; x \neq 0$$

The origin is asymptotically stable

If $xg(x) > 0$ for all $x \neq 0$, use

$$V(x) = \tfrac{1}{2}x^2 + \int_0^x g(y)\, dy$$

$$V(x) = \frac{1}{2}x^2 + \int_0^x g(y)\, dy$$

is positive definite for all x and radially unbounded since $V(x) \geq \frac{1}{2}x^2$

$$\dot{V}(x) = -xg(x) - g^2(x) < 0, \quad \forall\, x \neq 0$$

The origin is globally asymptotically stable

Example: Pendulum equation without friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\,a\sin x_1\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \tfrac{1}{2}x_2^2$$

$V(0)=0$ and $V(x)$ is positive definite over the domain $-2\pi < x_1 < 2\pi$

$$\dot{V}(x) = a\dot{x}_1\sin x_1 + x_2\dot{x}_2 = ax_2\sin x_1 - ax_2\sin x_1 = 0$$

The origin is stable

Since $\dot{V}(x)\equiv 0$, the origin is not asymptotically stable

Example: Pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\,a\sin x_1 - bx_2\end{aligned}$$

$$V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1\sin x_1 + x_2\dot{x}_2 = -\,bx_2^2$$

The origin is stable

$\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

The conditions of Lyapunov's theorem are only sufficient. Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable. It only means that such stability property cannot be established by using this Lyapunov function candidate

Try

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T P x + a(1 - \cos x_1) \\ &= \tfrac{1}{2}[x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1) \end{aligned}$$

$$p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

$$\begin{aligned}\dot{V}(x) &= (p_{11}x_1 + p_{12}x_2 + a\sin x_1)\,x_2 \\ &\quad + (p_{12}x_1 + p_{22}x_2)\,(-a\sin x_1 - bx_2) \\ &= a(1-p_{22})x_2\sin x_1 - ap_{12}x_1\sin x_1 \\ &\quad + (p_{11} - p_{12}b)\,x_1x_2 + (p_{12} - p_{22}b)\,x_2^2\end{aligned}$$

$p_{22} = 1, \quad p_{11} = bp_{12} \;\Rightarrow\; 0 < p_{12} < b,$ Take $p_{12} = b/2$

$$\dot{V}(x) = -\tfrac{1}{2}abx_1\sin x_1 - \tfrac{1}{2}bx_2^2$$

$$D = \{x \in R^2 \mid |x_1| < \pi\}$$

$V(x)$ is positive definite and $\dot{V}(x)$ is negative definite over D

The origin is asymptotically stable