

Chapter 4

Lyapunov Stability

The Invariance Principle

Time Varying Systems

Example: Pendulum equation with friction

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin x_1 - bx_2\end{aligned}$$

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

The origin is stable. $\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

However, near the origin, the solution cannot stay identically in the set $\{x_2 = 0\}$

Definitions: Let $x(t)$ be a solution of $\dot{x} = f(x)$

A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $\lim_{n \rightarrow \infty} t_n = \infty$, such that $x(t_n) \rightarrow p$ as $n \rightarrow \infty$

The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$; denoted by L^+

If $x(t)$ approaches an asymptotically stable equilibrium point \bar{x} , then \bar{x} is the positive limit point of $x(t)$ and $L^+ = \bar{x}$

A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle

A set M is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in \mathbb{R}$$

Examples:

- Equilibrium points
- Limit Cycles

A set M is a *positively invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

Example: The set $\Omega_c = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c

The distance from a point p to a set M is defined by

$$\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

$x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

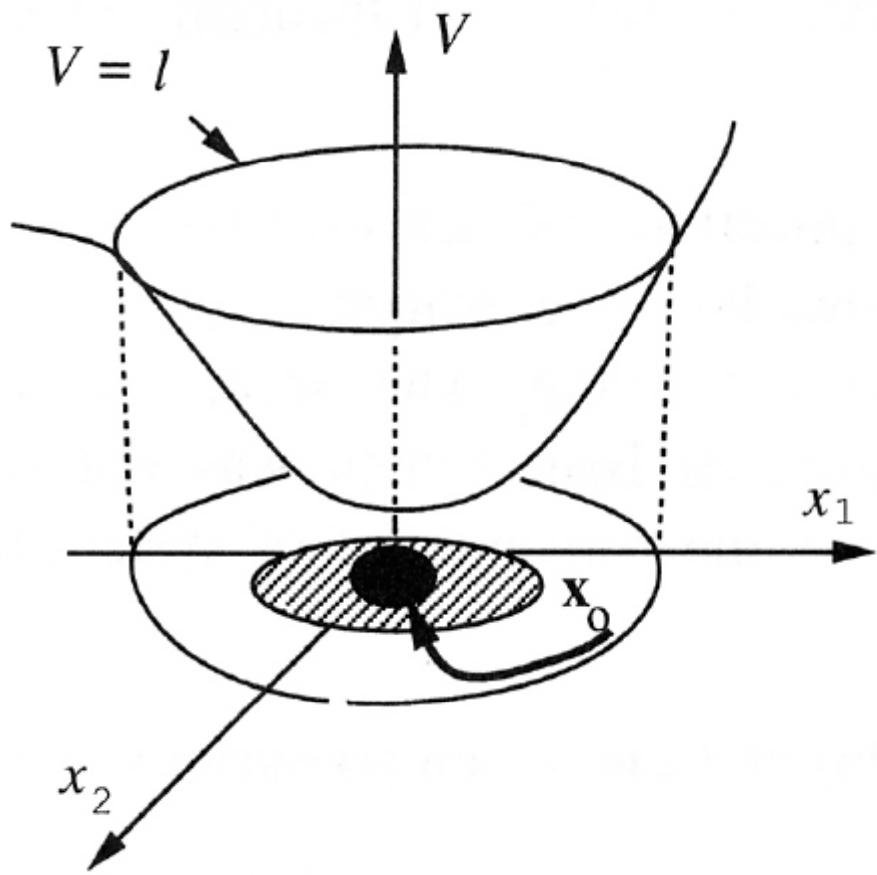
$$\text{dist}(x(t), M) < \varepsilon, \quad \forall t > T$$

Example: every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \rightarrow \infty$

Notice, however, that $x(t)$ does not converge to any specific point on the limit cycle

Lemma: If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \rightarrow \infty$

LaSalle's theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω . Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E . Then every solution starting in Ω approaches M as $t \rightarrow \infty$



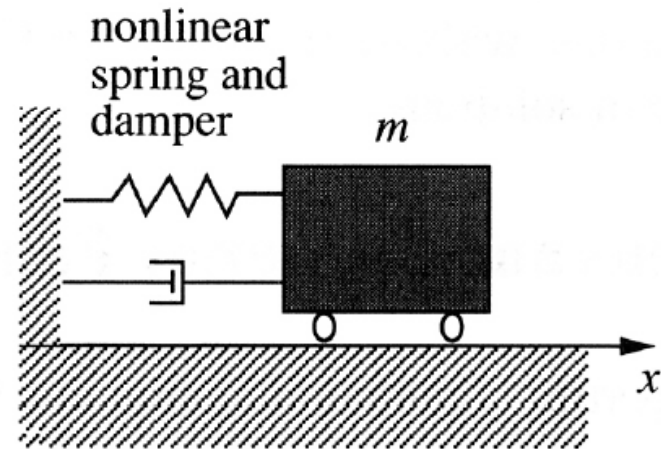
Ω ○

E ●

M ●

Example

$$m\ddot{x} + b\dot{x}|\dot{x}| + k_0 x + k_1 x^3 = 0$$



$$V(\mathbf{x}) = \frac{1}{2} m \dot{x}^2 + \int_0^x (k_0 x + k_1 x^3) dx = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k_0 x^2 + \frac{1}{4} k_1 x^4$$

- zero energy corresponds to the equilibrium point ($\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$)
- asymptotic stability implies the convergence of mechanical energy to zero
- instability is related to the growth of mechanical energy

$$\dot{V}(\mathbf{x}) = m \dot{x} \ddot{x} + (k_0 x + k_1 x^3) \dot{x} = \dot{x} (-b \dot{x} |\dot{x}|) = -b |\dot{x}|^3$$

$$E : \dot{V}(x) = -b|\dot{x}|^3 = 0 \Rightarrow \dot{x} = 0$$

$$M : (x, \dot{x}) = (0, 0)$$

- Assume M contains a point with a nonzero position x , then the acceleration at that

point is

$$\ddot{x} = -\frac{k_0}{m}x - \frac{k_1}{m}x^3 \neq 0$$

- This implies that the trajectory will move out of E . Not invariant. Contradiction!

Theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D . Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in S , other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable
- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction
- Furthermore, if $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$h_i(0) = 0, \quad yh_i(y) > 0, \quad \text{for } 0 < |y| < a$$

$$V(x) = \int_0^{x_1} h_1(y) dy + \frac{1}{2}x_2^2$$

$$D = \{-a < x_1 < a, \quad -a < x_2 < a\}$$

$$\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2h_2(x_2) \leq 0$$

$$\dot{V}(x) = 0 \Rightarrow x_2h_2(x_2) = 0 \Rightarrow x_2 = 0$$

$$S = \{x \in D \mid x_2 = 0\}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

The only solution that can stay identically in S is $x(t) \equiv 0$

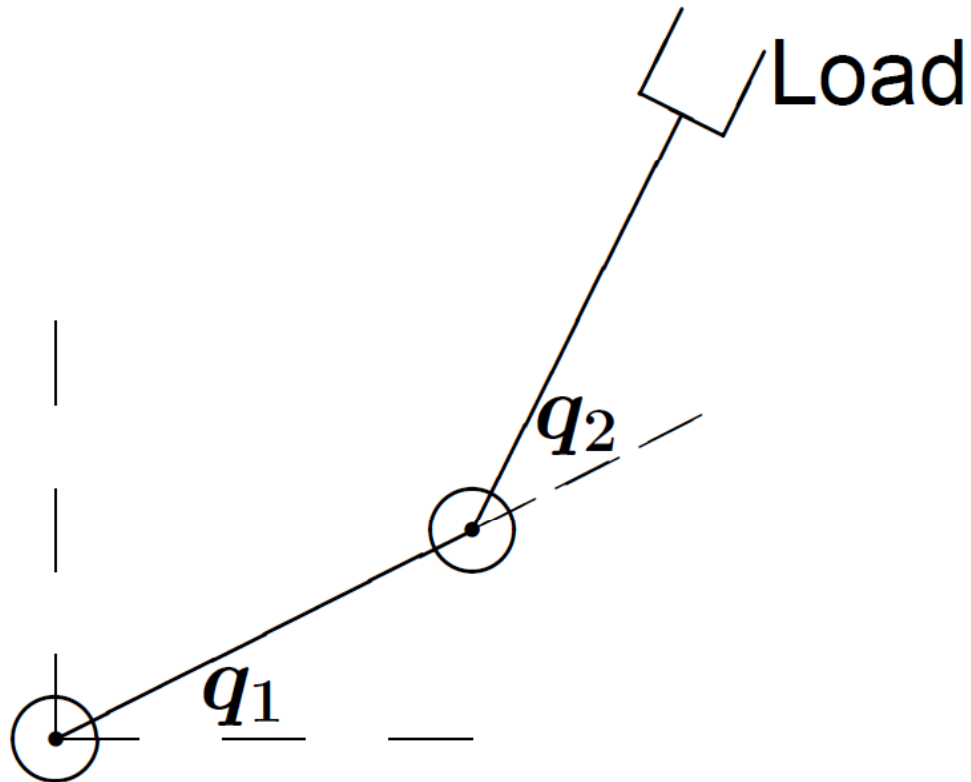
Thus, the origin is asymptotically stable

Suppose $a = \infty$ and $\int_0^y h_1(z) dz \rightarrow \infty$ as $|y| \rightarrow \infty$

Then, $D = \mathbb{R}^2$ and $V(x) = \int_0^{x_1} h_1(y) dy + \frac{1}{2}x_2^2$ is radially unbounded. $S = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$ and the only solution that can stay identically in S is $x(t) \equiv 0$

The origin is globally asymptotically stable

Example: m -link Robot Manipulator



Two-link Robot Manipulator

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$$

q is an m -dimensional vector of joint positions

u is an m -dimensional control (torque) inputs

$M = M^T > 0$ is the inertia matrix

$C(q, \dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces

$$(\dot{M} - 2C)^T = -(\dot{M} - 2C)$$

$D\dot{q}$ accounts for viscous damping; $D = D^T \geq 0$

$g(q)$ accounts for gravity forces; $g(q) = [\partial P(q)/\partial q]^T$

$P(q)$ is the total potential energy of the links due to gravity

Investigate the use of the (PD plus gravity compensation) control law

$$u = g(q) - K_p(q - q^*) - K_d \dot{q}$$

to stabilize the robot at a desired position q^* , where K_p and K_d are symmetric positive definite matrices

$$e = q - q^*, \quad \dot{e} = \dot{q}$$

$$\begin{aligned} M\ddot{e} &= M\ddot{q} \\ &= -C \dot{q} - D \dot{q} - g(q) + u \\ &= -C \dot{q} - D \dot{q} - K_p(q - q^*) - K_d \dot{q} \\ &= -C \dot{e} - D \dot{e} - K_p e - K_d \dot{e} \end{aligned}$$

$$M\ddot{e} = -C\dot{e} - D\dot{e} - K_p e - K_d \dot{e}$$

$$V = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T K_p e$$

$$\dot{V} = \dot{e}^T M\ddot{e} + \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e}$$

$$= -\dot{e}^T C\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e - \dot{e}^T K_d \dot{e} \\ + \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e}$$

$$= \frac{1}{2}\dot{e}^T (\dot{M} - 2C)\dot{e} - \dot{e}^T (K_d + D)\dot{e}$$

$$= -\dot{e}^T (K_d + D)\dot{e} \leq 0$$

$(K_d + D)$ is positive definite

$$\dot{V} = -\dot{e}^T (K_d + D) \dot{e} = 0 \Rightarrow \dot{e} = 0$$

$$M\ddot{e} = -C \dot{e} - D \dot{e} - K_p e - K_d \dot{e}$$

$$\dot{e}(t) \equiv 0 \Rightarrow \ddot{e}(t) \equiv 0 \Rightarrow K_p e(t) \equiv 0 \Rightarrow e(t) \equiv 0$$

By LaSalle's theorem the origin $(e = 0, \dot{e} = 0)$ is globally asymptotically stable

Converse Lyapunov Theorem–Exponential Stability

Let $x = 0$ be an exponentially stable equilibrium point for the system $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{\|x\| < r\}$. Let k , λ , and r_0 be positive constants with $r_0 < r/k$ such that

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall x(0) \in D_0, \quad \forall t \geq 0$$

where $D_0 = \{\|x\| < r_0\}$. Then, there is a continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \leq -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\|$$

for all $x \in D_0$, with positive constants c_1 , c_2 , c_3 , and c_4 .
Moreover, if f is continuously differentiable for all x , globally Lipschitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in R^n$.

Example: Consider the system $\dot{x} = f(x)$ where f is continuously differentiable in the neighborhood of the origin and $f(0) = 0$. Show that the origin is exponentially stable only if $A = [\partial f / \partial x](0)$ is Hurwitz

$$f(x) = Ax + G(x)x, \quad G(x) \rightarrow 0 \text{ as } x \rightarrow 0$$

Given any $L > 0$, there is $r_1 > 0$ such that

$$\|G(x)\| \leq L, \quad \forall \|x\| < r_1$$

Because the origin of $\dot{x} = f(x)$ is exponentially stable, let $V(x)$ be the function provided by the converse Lyapunov theorem over the domain $\{\|x\| < r_0\}$. Use $V(x)$ as a Lyapunov function candidate for $\dot{x} = Ax$

Converse Lyapunov Theorem–Asymptotic Stability

Let $x = 0$ be an asymptotically stable equilibrium point for $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \rightarrow \infty \text{ as } x \rightarrow \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall x \in R_A$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A .
When $R_A = \mathbb{R}^n$, $V(x)$ is radially unbounded

Time-varying Systems

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D$. The origin is an equilibrium point at $t = 0$ if

$$f(t, 0) = 0, \quad \forall t \geq 0$$

While the solution of the autonomous system

$$\dot{x} = f(x), \quad x(t_0) = x_0$$

depends only on $(t - t_0)$, the solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

may depend on both t and t_0

Comparison Functions

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$ is said to belong to class \mathcal{K} if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class \mathcal{K}_∞ if it defined for all $r \geq 0$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$
- A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$ is said to belong to class \mathcal{KL} if, for each fixed s , the mapping $\beta(r, s)$ belongs to class \mathcal{K} with respect to r and, for each fixed r , the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$

Example

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1+r^2) > 0$. It belongs to class \mathcal{K} , but not to class \mathcal{K}_∞ since $\lim_{r \rightarrow \infty} \alpha(r) = \pi/2 < \infty$
- $\alpha(r) = r^c$, for any positive real number c , is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r \rightarrow \infty} \alpha(r) = \infty$; thus, it belongs to class \mathcal{K}_∞
- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$. Hence, it belongs to class \mathcal{K}_∞

- $\beta(r, s) = r/(ksr + 1)$, for any positive real number k , is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr + 1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr + 1)^2} < 0$$

Moreover, $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$. Therefore, it belongs to class \mathcal{KL}

- $\beta(r, s) = r^c e^{-s}$, for any positive real number c , belongs to class \mathcal{KL}

Definition 4.4 *The equilibrium point $x = 0$ of (4.15) is*

- *stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq t_0 \geq 0 \quad (4.16)$$

- *uniformly stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, independent of t_0 , such that (4.16) is satisfied.*
- *unstable if it is not stable.*
- *asymptotically stable if it is stable and there is a positive constant $c = c(t_0)$ such that $x(t) \rightarrow 0$ as $t \rightarrow \infty$, for all $\|x(t_0)\| < c$.*

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class \mathcal{K} function α and a positive constant c , independent of t_0 , such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class \mathcal{KL} function β and a positive constant c , independent of t_0 , such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0), \quad \forall t \geq t_0 \geq 0, \quad \forall \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

- exponentially stable if there exist positive constants c , k , and λ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0)\| < c$$

- globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

Derivative of V

$$\begin{aligned}\frac{dV}{dt} &= \lim_{\Delta t \rightarrow \infty} \frac{V(x(t + \Delta t), t + \Delta t) - V(x(t), t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow \infty} \frac{V(x(t + \Delta t), t + \Delta t) - V(x(t + \Delta t), t) + V(x(t + \Delta t), t) - V(x(t), t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow \infty} \frac{V(x(t + \Delta t), t + \Delta t) - V(x(t + \Delta t), t)}{\Delta t} + \lim_{\Delta t \rightarrow \infty} \frac{V(x(t + \Delta t), t) - V(x(t), t)}{\Delta t} \\ &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t)\end{aligned}$$

Theorem: Let the origin $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

$$(1) \quad W_1(x) \leq V(t, x) \leq W_2(x)$$

$$(2) \quad \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D . Then, the origin is uniformly stable

Theorem: Suppose the assumptions of the previous theorem are satisfied with

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D . Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r \mid W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0$$

for some class \mathcal{KL} function β . Finally, if $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable

Theorem: Suppose the assumptions of the previous theorem are satisfied with

$$k_1 \|x\|^a \leq V(t, x) \leq k_2 \|x\|^a$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -k_3 \|x\|^a$$

for all $t \geq 0$ and $x \in D$, where k_1, k_2, k_3 , and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable.

A function $V(t, x)$ is said to be *positive semidefinite* if $V(t, x) \geq 0$. It is said to be *positive definite* if $V(t, x) \geq W_1(x)$ for some positive definite function $W_1(x)$, *radially unbounded* if $W_1(x)$ is so, and *decreascent* if $V(t, x) \leq W_2(x)$. A function $V(t, x)$ is said to be *negative definite* (*semidefinite*) if $-V(t, x)$ is *positive definite* (*semidefinite*). Therefore, Theorems 4.8 and 4.9 say that *the origin is uniformly stable if there is a continuously differentiable, positive definite, decreascent function $V(t, x)$, whose derivative along the trajectories of the system is negative semidefinite. It is uniformly asymptotically stable if the derivative is negative definite, and globally uniformly asymptotically stable if the conditions for uniform asymptotic stability hold globally with a radially unbounded $V(t, x)$.*

Example:

$$\dot{x} = -[1 + g(t)]x^3, \quad g(t) \geq 0, \quad \forall t \geq 0$$

$$V(x) = \frac{1}{2}x^2$$

$$\dot{V}(t, x) = -[1 + g(t)]x^4 \leq -x^4, \quad \forall x \in \mathbf{R}, \quad \forall t \geq 0$$

The origin is globally uniformly asymptotically stable

Example:

$$\dot{x}_1 = -x_1 - g(t)x_2$$

$$\dot{x}_2 = x_1 - x_2$$

$$0 \leq g(t) \leq k \quad \text{and} \quad \dot{g}(t) \leq g(t), \quad \forall t \geq 0$$

$$V(t, x) = x_1^2 + [1 + g(t)]x_2^2$$

$$x_1^2 + x_2^2 \leq V(t, x) \leq x_1^2 + (1 + k)x_2^2, \quad \forall x \in \mathbb{R}^2$$

$$\dot{V}(t, x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

$$2 + 2g(t) - \dot{g}(t) \geq 2 + 2g(t) - g(t) \geq 2$$

$$\dot{V}(t, x) \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x$$

The origin is globally exponentially stable

$$\lambda_{\min}(P)x^T x \leq x^T P x \leq \lambda_{\max}(P)x^T x$$