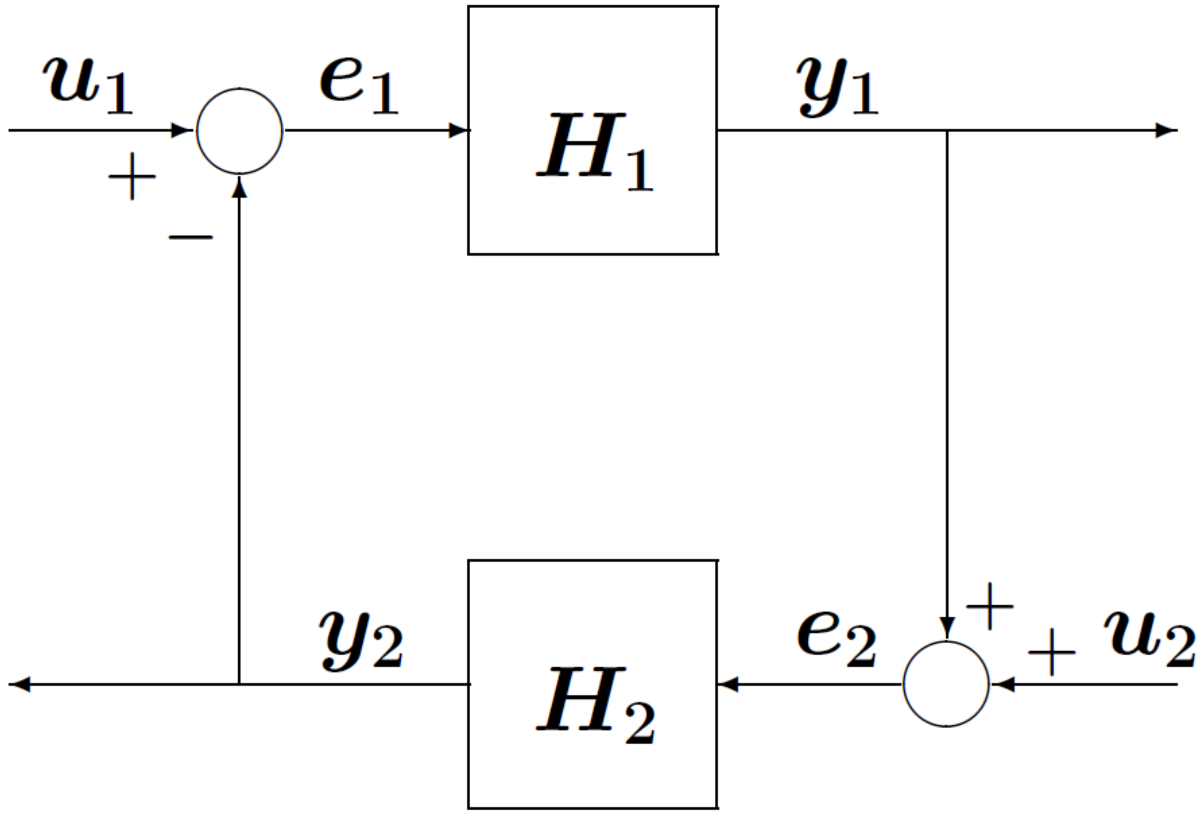
Chapter 6
Passivity

Feedback Systems:

Passivity Theorems



$$\dot{x}_i = f_i(x_i, e_i), \qquad y_i = h_i(x_i, e_i)$$

$$y_i = h_i(t, e_i)$$

State Models

Let us now define passivity for a dynamical system represented by the state model

$$\dot{x} = f(x, u) \tag{6.6}$$

$$y = h(x, u) \tag{6.7}$$

where $f : R^n \times R^p \to R^n$ is locally Lipschitz, $h : R^n \times R^p \to R^p$ is continuous, $f(0,0) = 0$, and $h(0,0) = 0$. The system has the same number of inputs and outputs. The following RLC circuit motivates the definition.

- The origin is an equilibrium point.

Passivity Theorems

Theorem 6.1: The feedback connection of two passive systems is passive

Theorem 6.3: Consider the feedback connection of two dynamical systems. When $u = 0$, the origin of the closed-loop system is asymptotically stable if each feedback component is either

- strictly passive, or
- output strictly passive and zero-state observable

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable

Theorem 6.4: Consider the feedback connection of a strictly passive dynamical system with a passive memoryless function. When $u = 0$, the origin of the closed-loop system is uniformly asymptotically stable. if the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable

Prove using $V = V_1 + V_2$ as a Lyapunov function candidate

Proof of Theorem 6.3: H_1 is SP; H_2 is OSP & ZSO

$$e_1^T y_1 \ge \dot{V}_1 + \psi_1(x_1), \qquad \psi_1(x_1) > 0, \ \forall \ x_1 \neq 0$$

$$e_2^T y_2 \ge \dot{V}_2 + y_2^T \rho_2(y_2), \qquad y_2^T \rho(y_2) > 0, \ \forall y_2 \neq 0$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$V(x) = V_1(x_1) + V_2(x_2)$$

$$\dot{V} \le u^T y - \psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$u = 0 \;\Rightarrow\; \dot{V} \le -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$\dot{V} = 0 \;\Rightarrow\; x_1 = 0 \text{ and } y_2 = 0$$

$$y_2(t) \equiv 0 \;\Rightarrow\; e_1(t) \equiv 0 \;\; (\,\&\; x_1(t) \equiv 0) \;\Rightarrow\; y_1(t) \equiv 0$$

$$y_1(t) \equiv 0 \;\Rightarrow\; e_2(t) \equiv 0$$

By zero-state observability of H_2: $y_2(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0$

Apply the invariance principle

Example

$$\underbrace{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + e_1 \\ y_1 &= x_2 + e_1 \end{aligned}}_{H_1} \qquad \Bigg| \qquad \underbrace{\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -bx_3 - x_4^3 + e_2 \\ y_2 &= x_4 \end{aligned}}_{H_2}$$

$$a,\ b,\ k > 0$$

$$V_1 = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2$$

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1$$

With $e_1 = 0, \quad y_1(t) \equiv 0 \ \Leftrightarrow \ x_2(t) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0$

H_1 is output strictly passive and zero-state observable

$$V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$$

$$\dot{V}_2 = bx_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

With $e_2 = 0, \quad y_2(t) \equiv 0 \;\Leftrightarrow\; x_4(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$

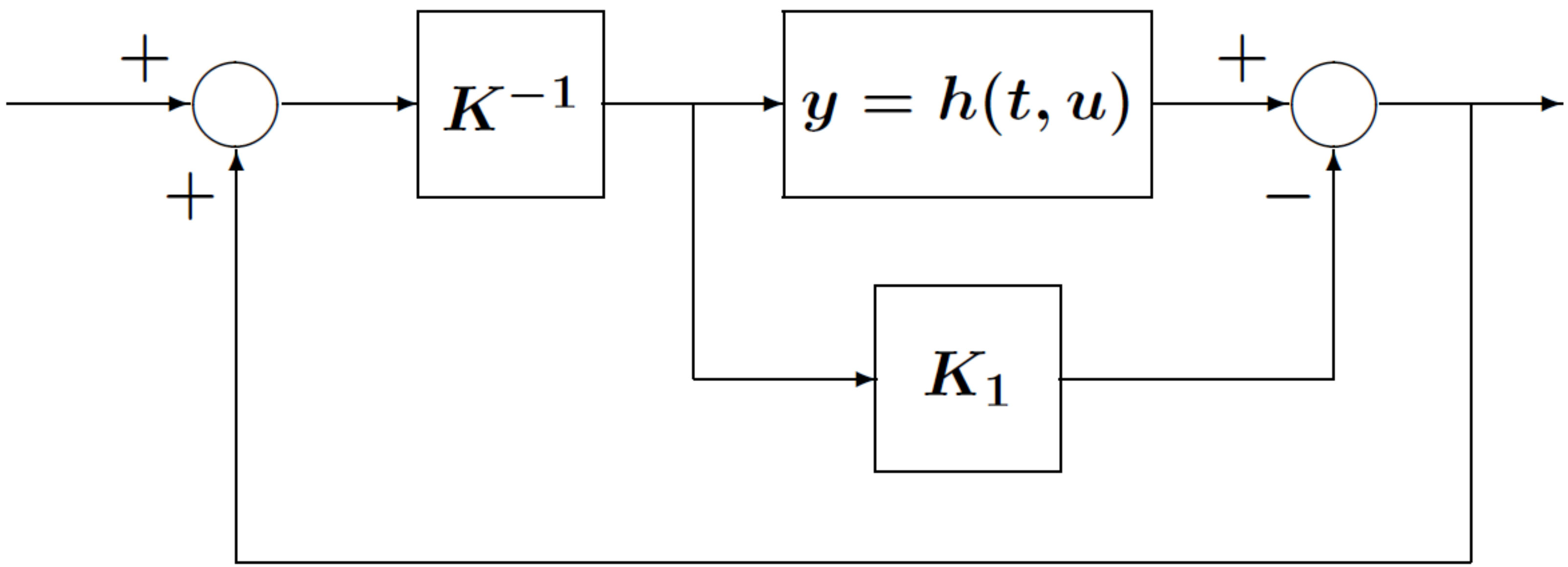
H_2 is output strictly passive and zero-state observable

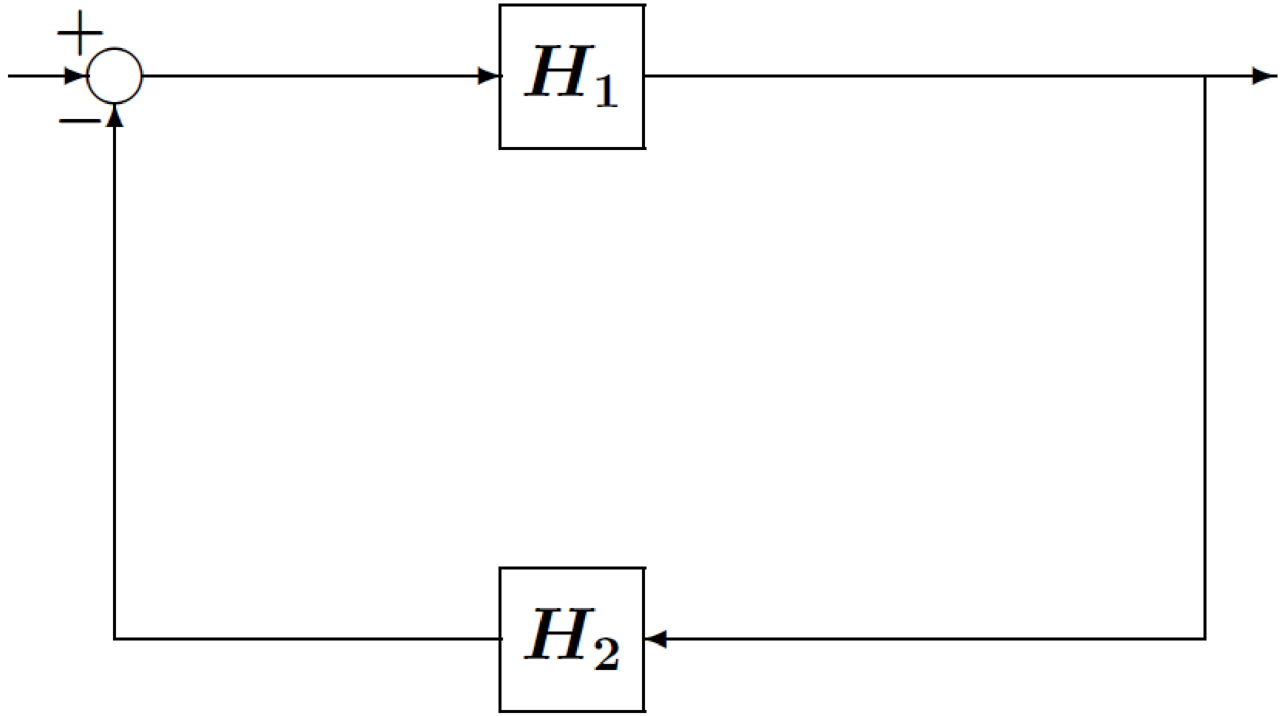
V_1 and V_2 are radially unbounded

The origin is globally asymptotically stable

Loop Transformations

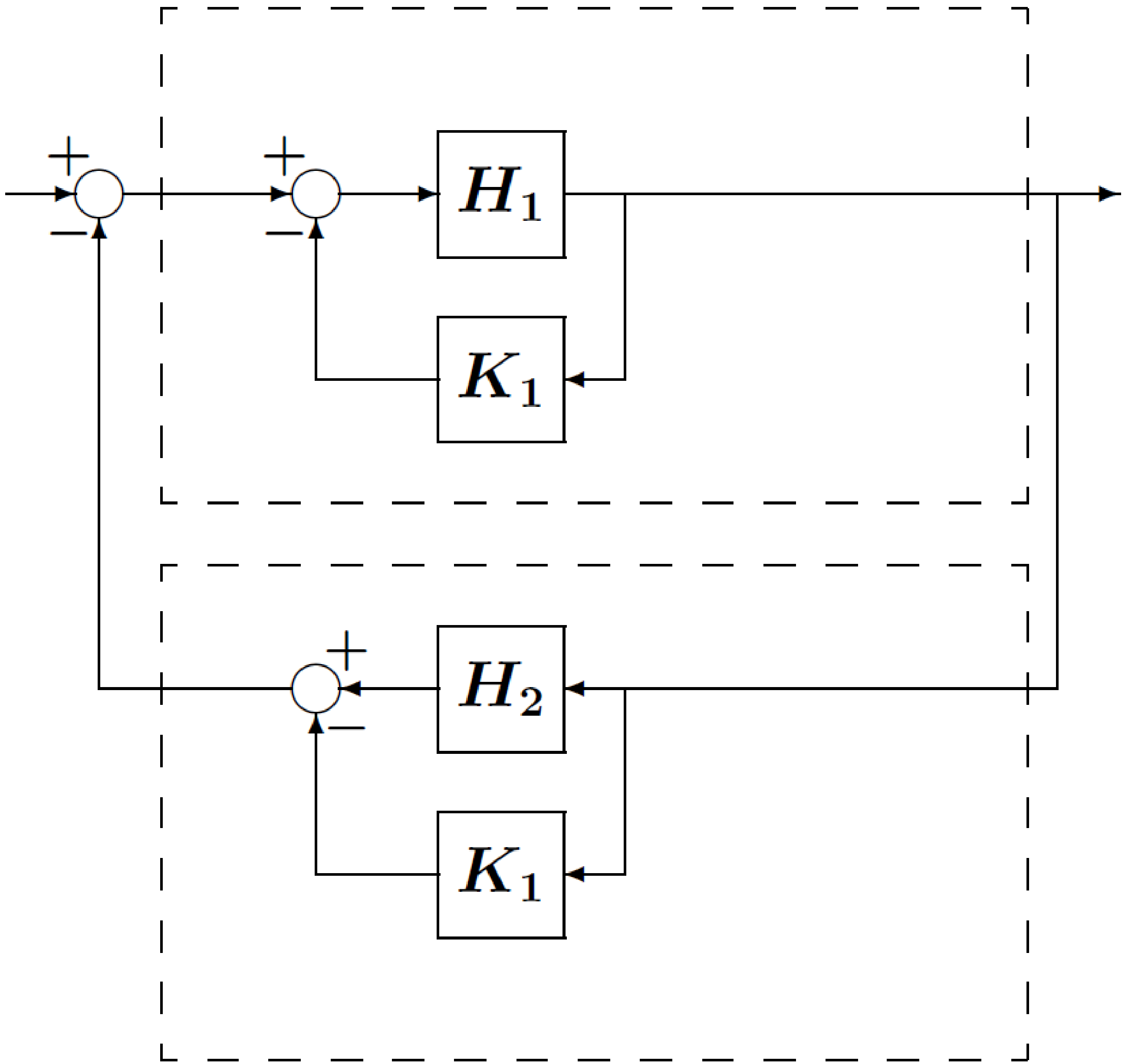
Recall that a memoryless function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback

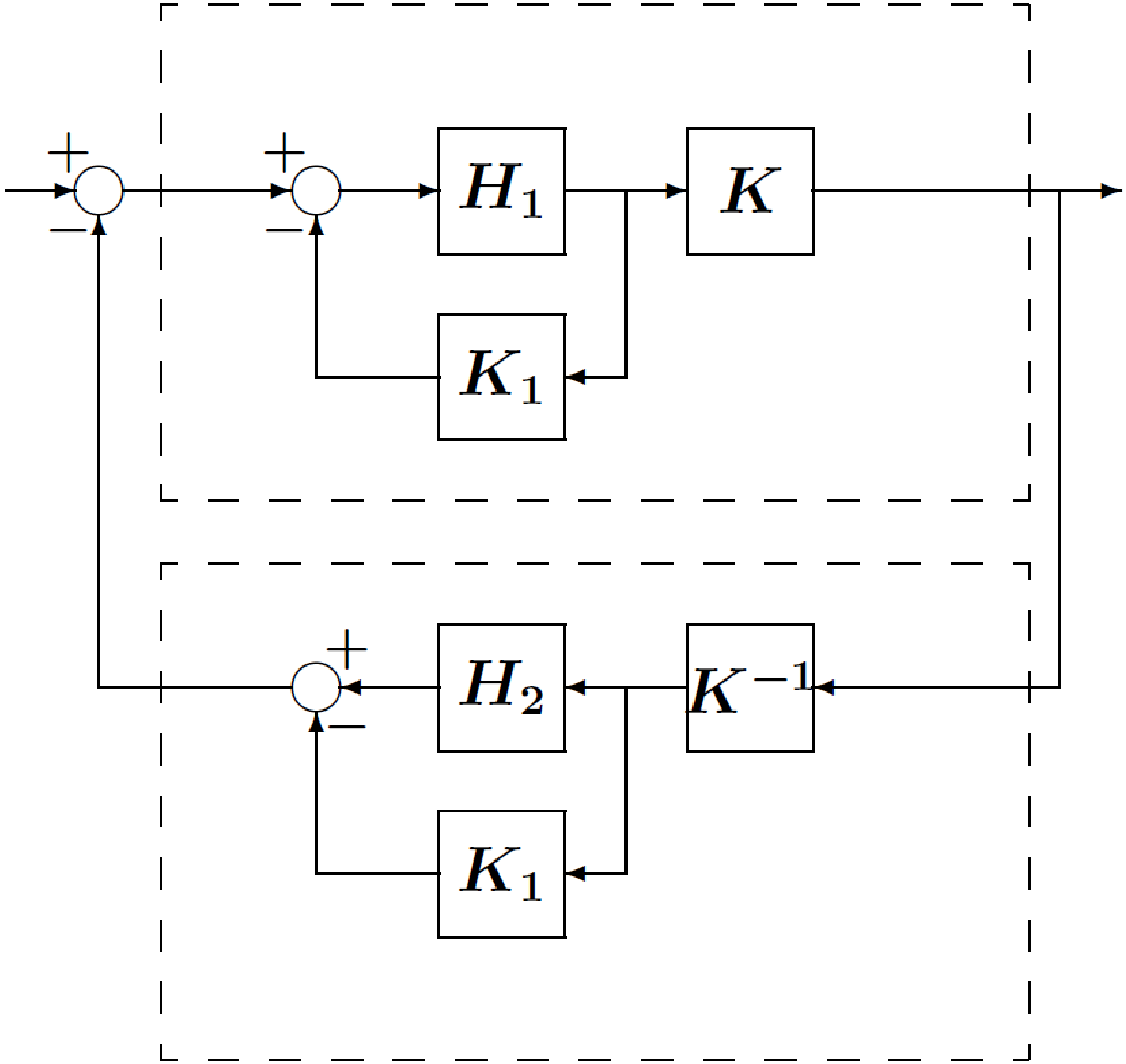




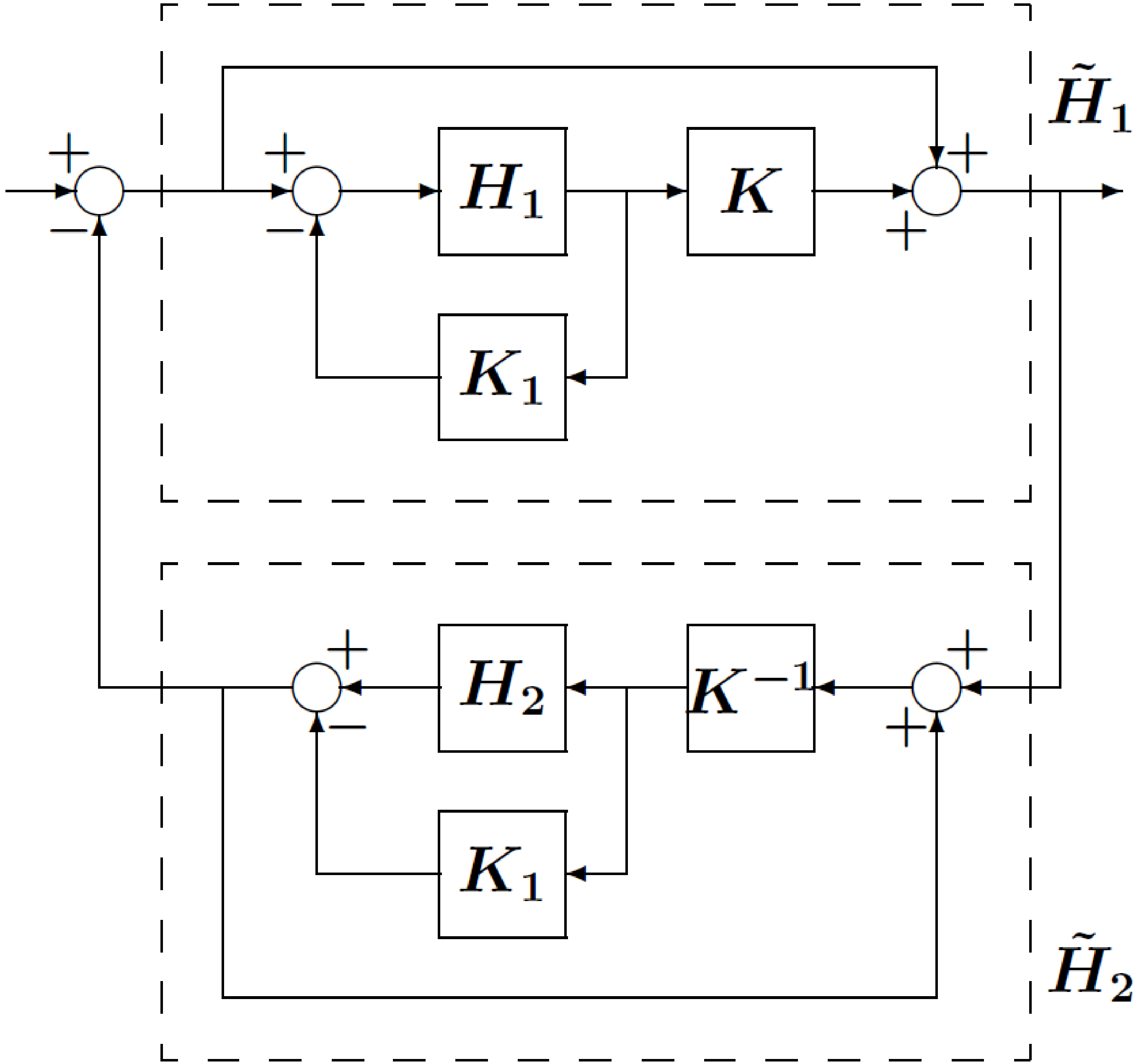
H_1 is a dynamical system

H_2 is a memoryless function in the sector $[K_1, K_2]$


+
+
H1
K1
+
H2
K1
−
−
−



+
−
+
−
H_1
K
K_1
+
−
H_2
K^{-1}
K_1


$\tilde{H}_1$
H_1
K
K_1
H_2
K^{-1}
K_1
$\tilde{H}_2$

Example

$$\underbrace{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) + bx_2 + e_1 \\ y_1 &= x_2 \end{aligned}}_{H_1} \quad \Bigg| \quad \underbrace{y_2 = \sigma(e_2)}_{H_2}$$

$$\sigma \in [\alpha, \beta], \;\; h \in [\alpha_1, \infty], \;\; b > 0, \;\; \alpha_1 > 0, \;\; k = \beta - \alpha > 0$$

$$\underbrace{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 + \tilde{e}_1 \\ \tilde{y}_1 &= kx_2 + \tilde{e}_1 \end{aligned}}_{\tilde{H}_1} \quad \Bigg| \quad \underbrace{\tilde{y}_2 = \tilde{\sigma}(\tilde{e}_2)}_{\tilde{H}_2}$$

$$\tilde{\sigma} \in [0, \infty], \qquad a = \alpha - b$$

Assume $a = \alpha - b > 0$ and show that $\tilde{H}_1$ is strictly passive

$$V_1 = k \int_0^{x_1} h(s)\, ds + x^T P x$$

$$V_1 = k \int_0^{x_1} h(s)\, ds + p_{11} x_1^2 + 2 p_{12} x_1 x_2 + p_{22} x_2^2$$

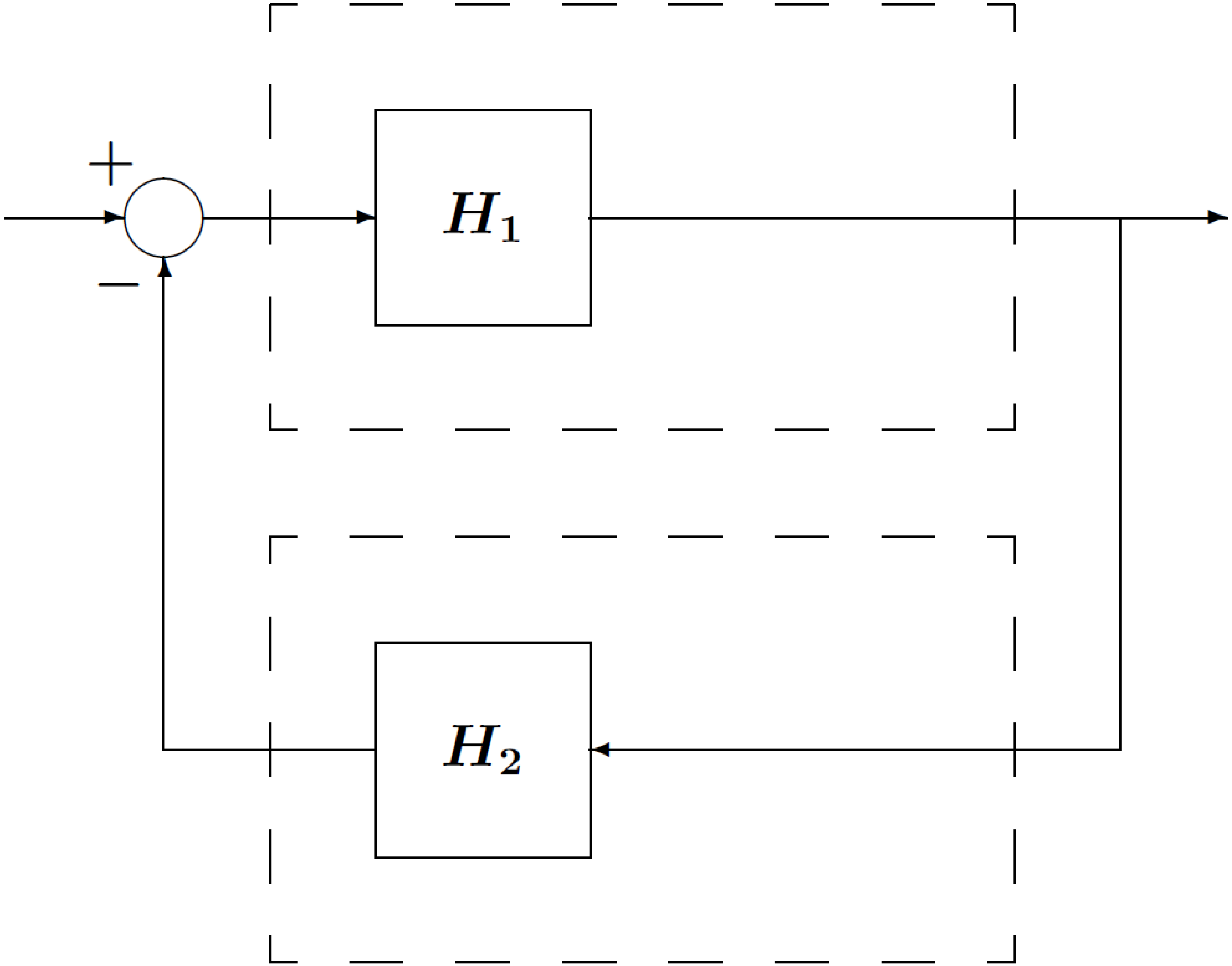
$$\begin{aligned}\dot{V} &= kh(x_1)x_2 + 2(p_{11}x_1 + p_{12}x_2)x_2 \\ &\quad 2(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + \tilde{e}_1]\end{aligned}$$

Take $p_{22} = k/2, \quad p_{11} = ap_{12}$

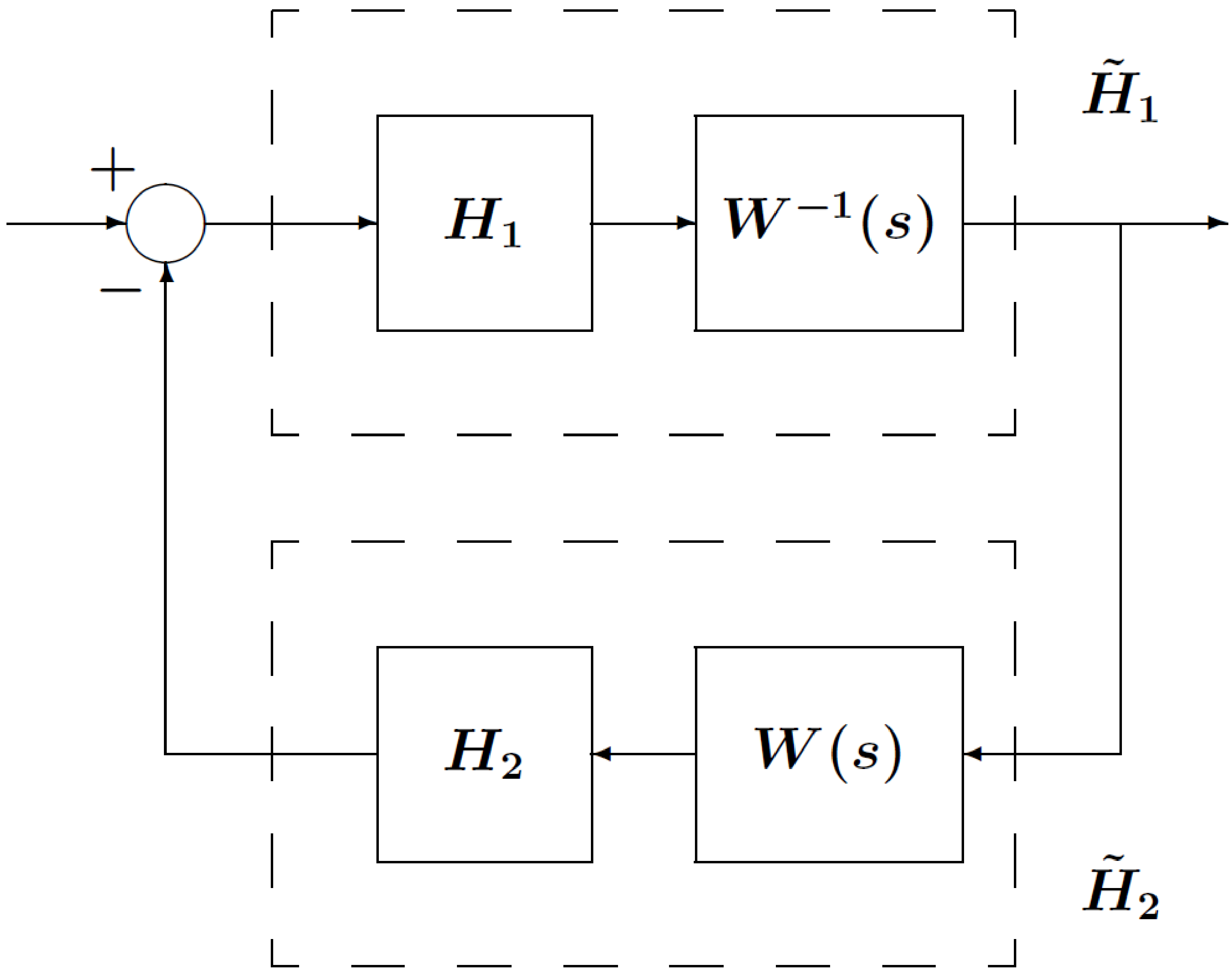
$$
\begin{aligned}
\dot{V} &= -2p_{12}x_1h(x_1) - (ka - 2p_{12})x_2^2 \\
&\quad + kx_2\tilde{e}_1 + 2p_{12}x_1\tilde{e}_1 \\
&= -2p_{12}x_1h(x_1) - (ka - 2p_{12})x_2^2 \\
&\quad + (kx_2 + \tilde{e}_1)\tilde{e}_1 - \tilde{e}_1^2 + 2p_{12}x_1\tilde{e}_1 \\
\tilde{y}_1\tilde{e}_1 &= \dot{V} + 2p_{12}x_1h(x_1) + (ka - 2p_{12})x_2^2 \\
&\quad + (\tilde{e}_1 - p_{12}x_1)^2 - p_{12}^2x_1^2 \\
&\geq \dot{V} + p_{12}(2\alpha_1 - p_{12})x_1^2 + (ka - 2p_{12})x_2^2
\end{aligned}
$$

Take $0 < p_{12} < \min\left\{\frac{ak}{2},\ 2\alpha_1\right\} \Rightarrow p_{12}^2 < 2p_{12}\frac{k}{2} = p_{11}p_{22}$

$\tilde{H}_1$ is strictly passive. By Theorem 6.4 the origin is globally asymptotically stable (when $u = 0$)



+
−
H_1
H_2



+
−
H_1
$W^{-1}(s)$
$\tilde{H}_1$
H_2
$W(s)$
$\tilde{H}_2$

Example

$$H_1: \quad \dot{x} = Ax + Be_1, \quad y_1 = Cx$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$H_2: \quad y_2 = h(e_2), \quad h \in [0, \infty]$$

$$C(sI - A)^{-1}B = \frac{1}{(s^2 + s + 1)} \qquad \text{Not PR}$$

$$W(s) = \frac{1}{as + 1} \quad \Rightarrow \quad \frac{(as + 1)}{(s^2 + s + 1)}$$

$$\tilde{H}_1: \quad \dot{x} = Ax + Be_1, \quad \tilde{y}_1 = \tilde{C}x = \begin{bmatrix} 1 & a \end{bmatrix} x$$

$$\frac{(as+1)}{(s^2+s+1)}$$

$$Re\left[\frac{1+j\omega a}{1-\omega^2+j\omega}\right] = \frac{1+(a-1)\omega^2}{(1-\omega^2)^2+\omega^2}$$

$$\lim_{\omega\to\infty} \omega^2 Re\left[\frac{1+j\omega a}{1-\omega^2+j\omega}\right] = a-1$$

$$a > 1 \quad \Rightarrow \quad \frac{(as+1)}{(s^2+s+1)} \text{ is SPR}$$

$$V_1 = \tfrac{1}{2}x^T P x, \quad PA + A^T P = -L^T L - \varepsilon P, \;\; PB = \tilde{C}^T$$

$$\tilde{H}_2: \quad a\dot{e}_2 = -e_2 + \tilde{e}_2, \qquad y_2 = h(e_2), \quad h \in [0, \infty]$$

$\tilde{H}_2$ is strictly passive with $V_2 = a\int_0^{e_2} h(s)\, ds$. Use

$$V = V_1 + V_2 = \frac{1}{2}x^T P x + a\int_0^{e_2} h(s)\, ds$$

as a Lyapunov function candidate for the original feedback connection

$$
\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T P x + ah(e_2)\dot{e}_2 \\
&= \tfrac{1}{2}x^T P[Ax - Bh(e_2)] + \tfrac{1}{2}[Ax - Bh(e_2)]^T P x \\
&\quad + ah(e_2)C[Ax - Bh(e_2)] \\
&= -\tfrac{1}{2}x^T L^T L x - (\varepsilon/2)x^T P x - x^T \tilde{C}^T h(e_2) \\
&\quad + ah(e_2)CAx \\
&= -\tfrac{1}{2}x^T L^T L x - (\varepsilon/2)x^T P x \\
&\quad - x^T[C + aCA]^T h(e_2) + ah(e_2)CAx \\
&= -\tfrac{1}{2}x^T L^T L x - (\varepsilon/2)x^T P x - e_2^T h(e_2) \\
&\leq -(\varepsilon/2)x^T P x
\end{aligned}
$$

The origin is globally asymptotically stable