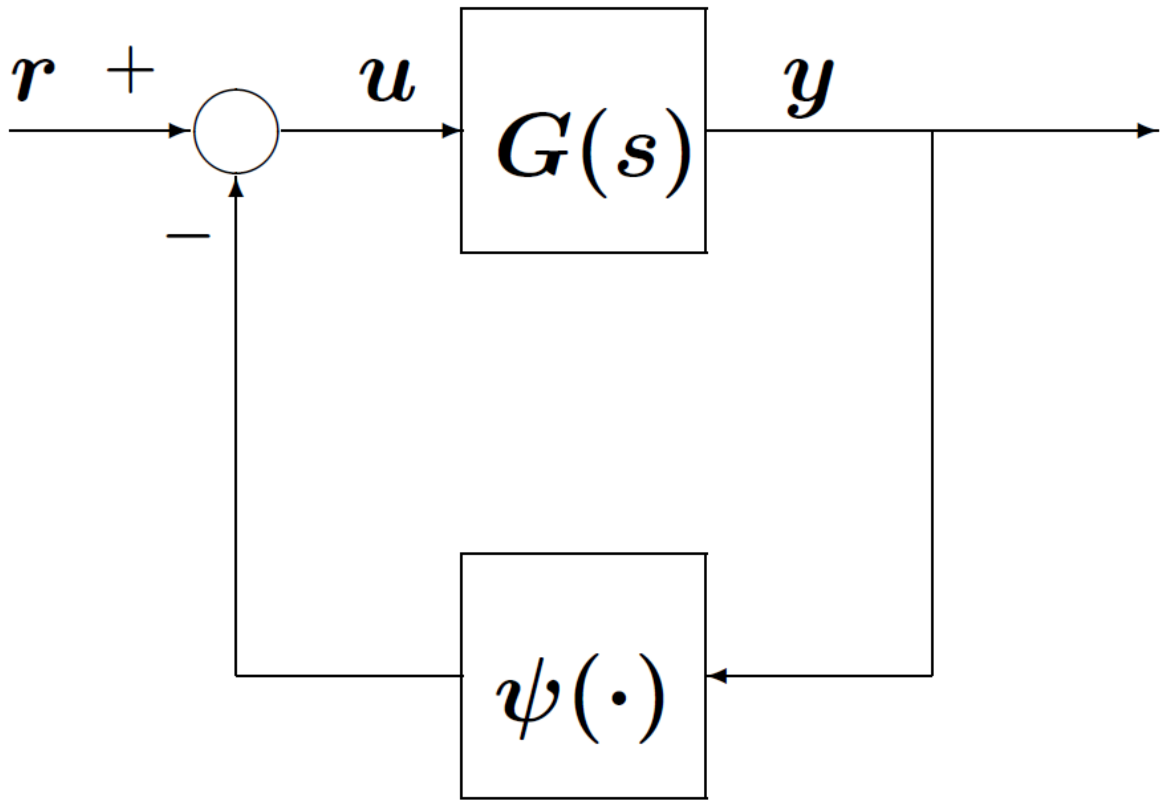
# Chapter 7
# Frequency Domain Analysis of Feedback Systems

Circle Criterion

Popov Criterion




The system is absolutely stable if (when $r = 0$) the origin is globally asymptotically stable for all memoryless time-invariant nonlinearities in a given sector

# Circle Criterion

Suppose $G(s) = C(sI - A)^{-1}B + D$ is SPR and $\psi \in [0, \infty]$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(y) \end{aligned}$$

By the KYP Lemma, $\exists\ P = P^T > 0,\ L, W, \varepsilon > 0$

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

$$V(x) = \tfrac{1}{2} x^T P x$$

$$
\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T Px \\
&= \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu \\
&= -\tfrac{1}{2}x^T L^T Lx - \tfrac{1}{2}\varepsilon x^T Px + x^T(C^T - L^T W)u \\
&= -\tfrac{1}{2}x^T L^T Lx - \tfrac{1}{2}\varepsilon x^T Px + (Cx + Du)^T u \\
&\quad - u^T Du - x^T L^T Wu
\end{aligned}
$$

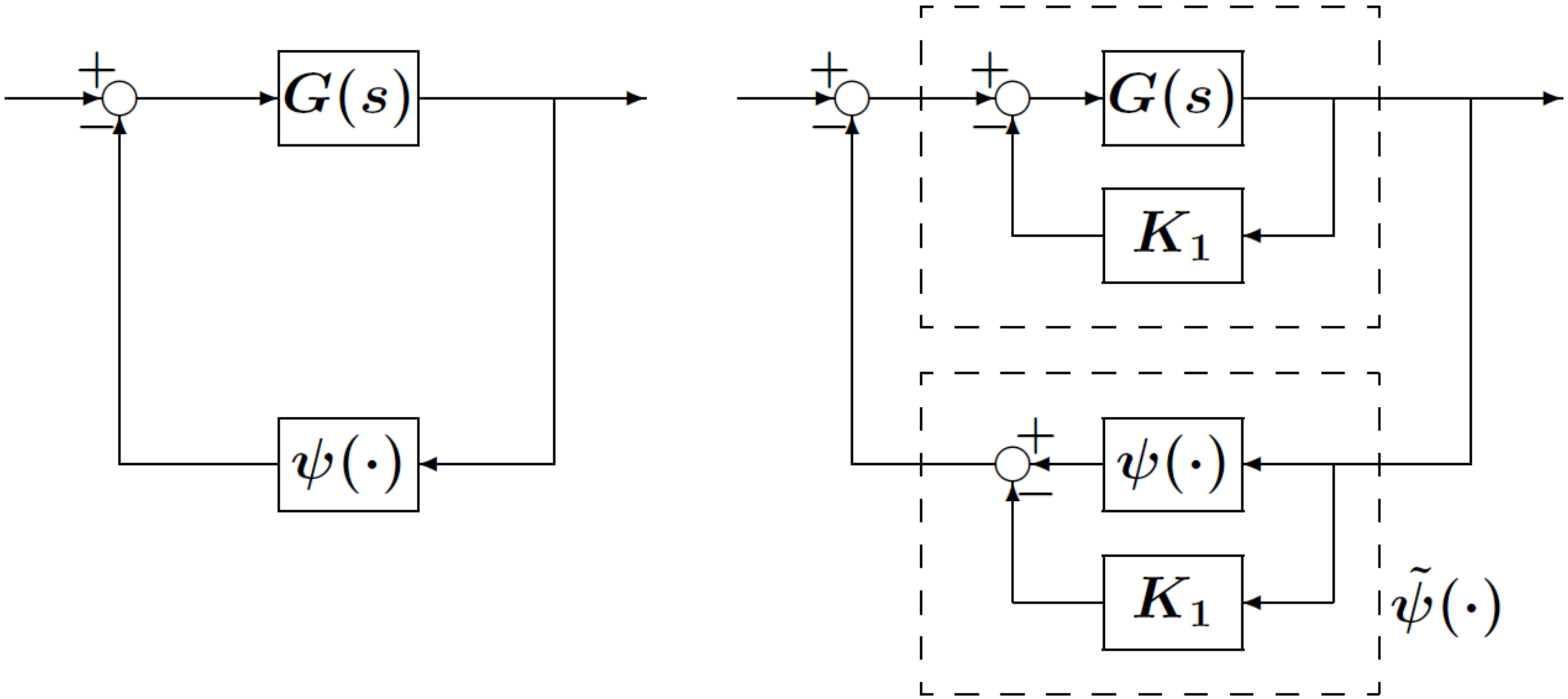
$$
u^T Du = \tfrac{1}{2}u^T(D + D^T)u = \tfrac{1}{2}u^T W^T Wu
$$

$$
\dot{V} = -\tfrac{1}{2}\varepsilon x^T Px - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) - y^T\psi(y)
$$

$$
y^T\psi(y) \geq 0 \quad \Rightarrow \quad \dot{V} \leq -\tfrac{1}{2}\varepsilon x^T Px
$$

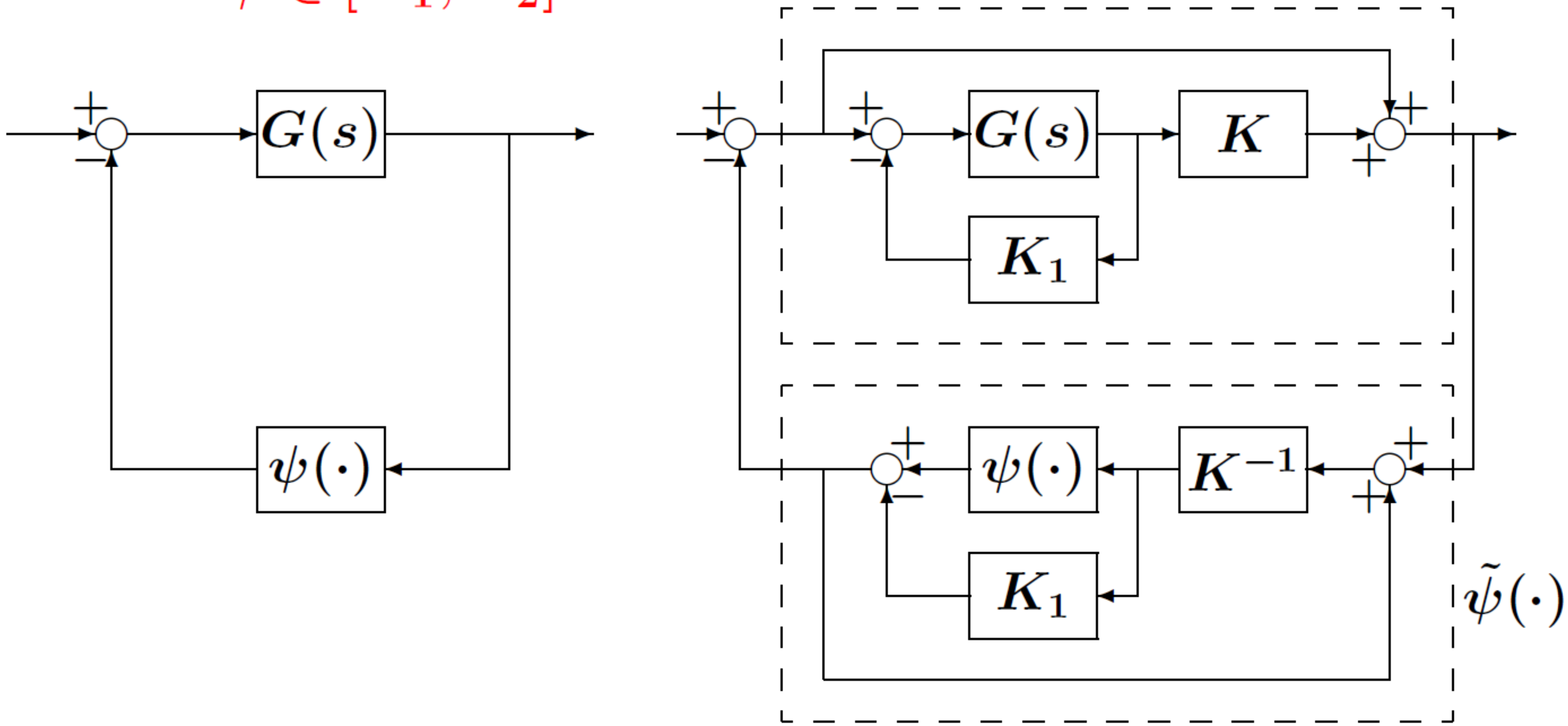
The origin is globally exponentially stable

What if $\psi \in [K_1, \infty]$?




$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $G(s)[I + K_1 G(s)]^{-1}$ is SPR

## What if $\psi \in [K_1, K_2]$?




$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $I + KG(s)[I + K_1 G(s)]^{-1}$ is SPR

$$I + KG(s)[I + K_1 G(s)]^{-1} = [I + K_2 G(s)][I + K_1 G(s)]^{-1}$$

Theorem (Circle Criterion): The system is absolutely stable if

- $\psi \in [K_1, \infty]$ and $G(s)[I + K_1 G(s)]^{-1}$ is SPR, or
- $\psi \in [K_1, K_2]$ and $[I + K_2 G(s)][I + K_1 G(s)]^{-1}$ is SPR

Scalar Case: $\psi \in [\alpha, \beta], \; \beta > \alpha$

The system is absolutely stable if

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} \quad \text{is Hurwitz and}$$

$$\text{Re}\left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)}\right] > 0, \;\; \forall\, \omega \in [0, \infty]$$

Case 1: $\alpha > 0$

By the Nyquist criterion

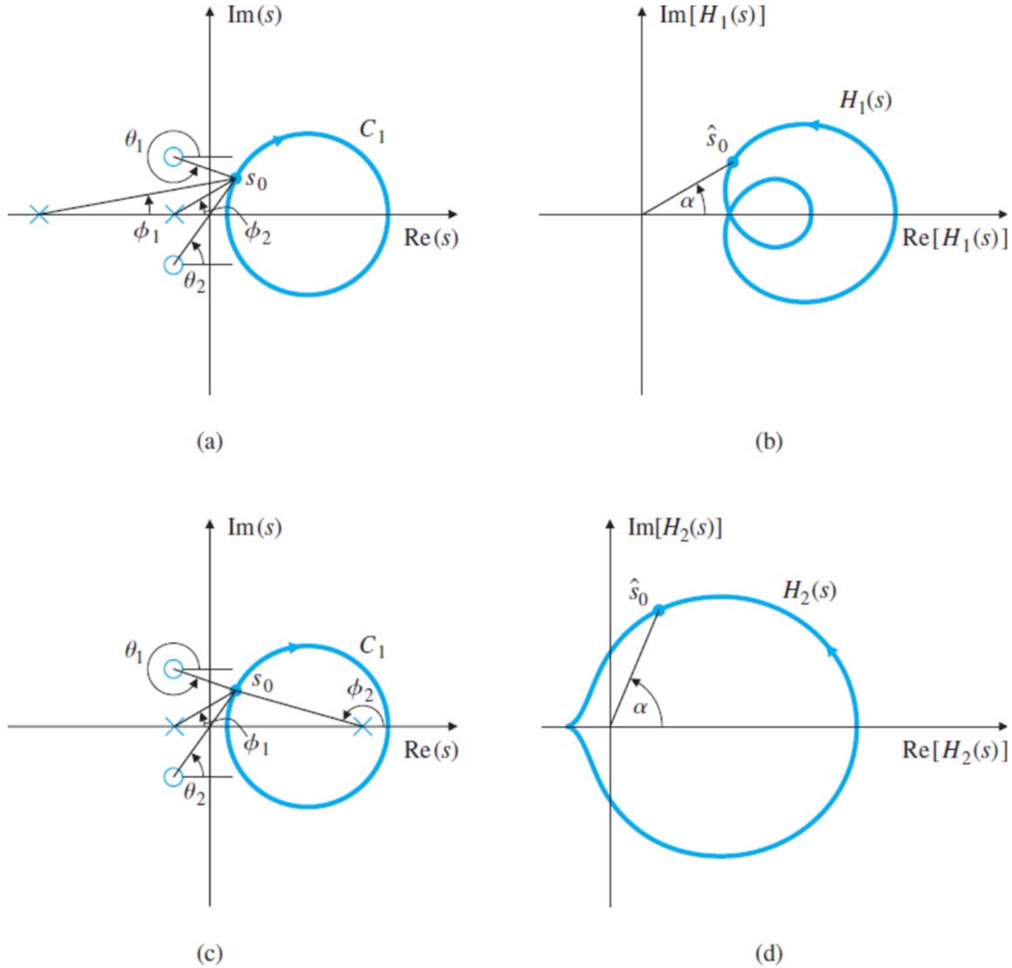
$$\frac{1+\beta G(s)}{1+\alpha G(s)} = \frac{1}{1+\alpha G(s)} + \frac{\beta G(s)}{1+\alpha G(s)}$$

is Hurwitz if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha) + j0$ and encircles it $m$ times in the counterclockwise direction, where $m$ is the number of poles of $G(s)$ in the open right-half complex plane

$$\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)} = \frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)}$$

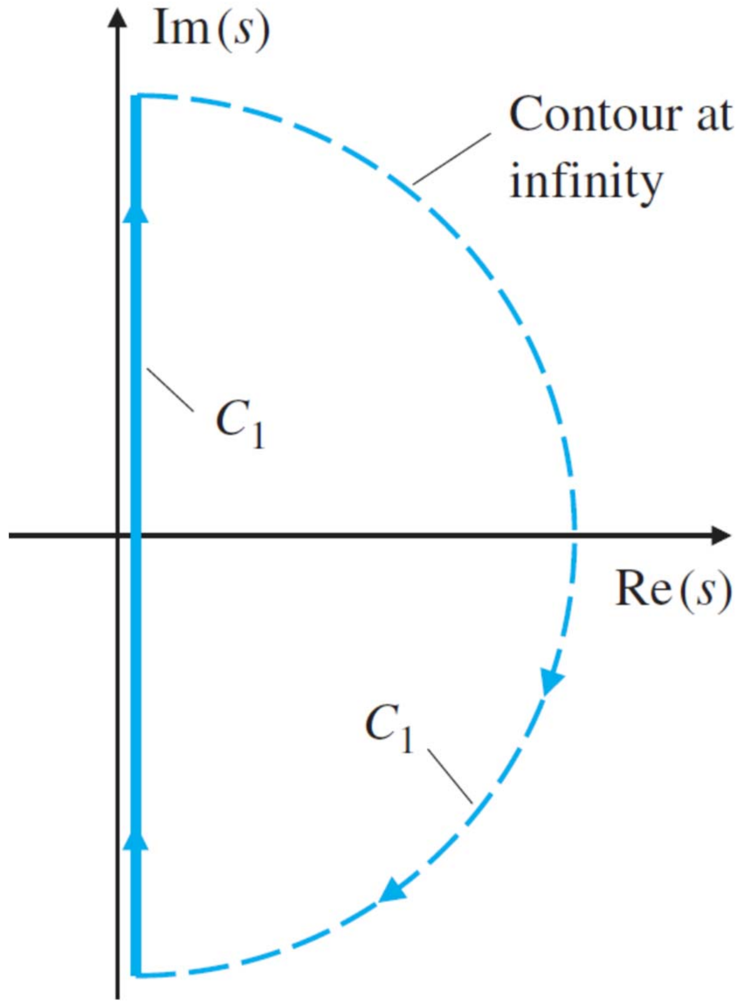
# Nyquist Criterion

- Principle of Argument
- N=Z-P

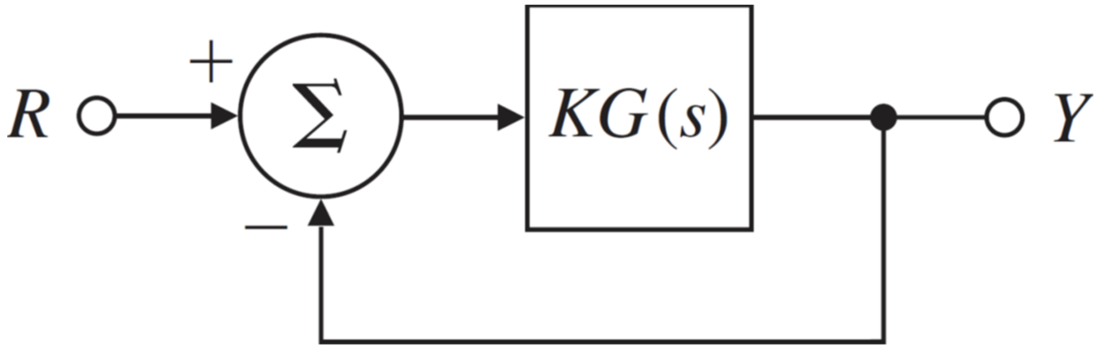


# Nyquist Criterion

- Nyquist Path
- N=Z-P



# Nyquist Criterion

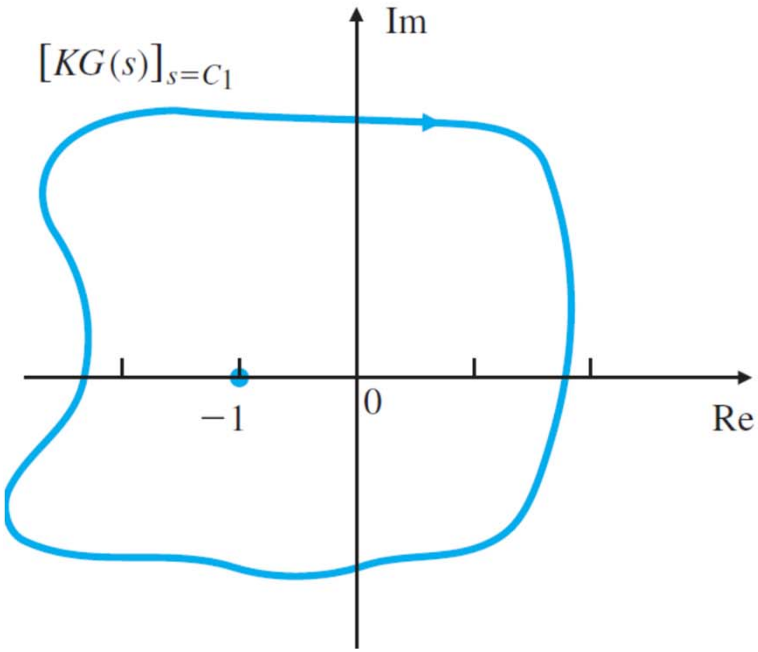


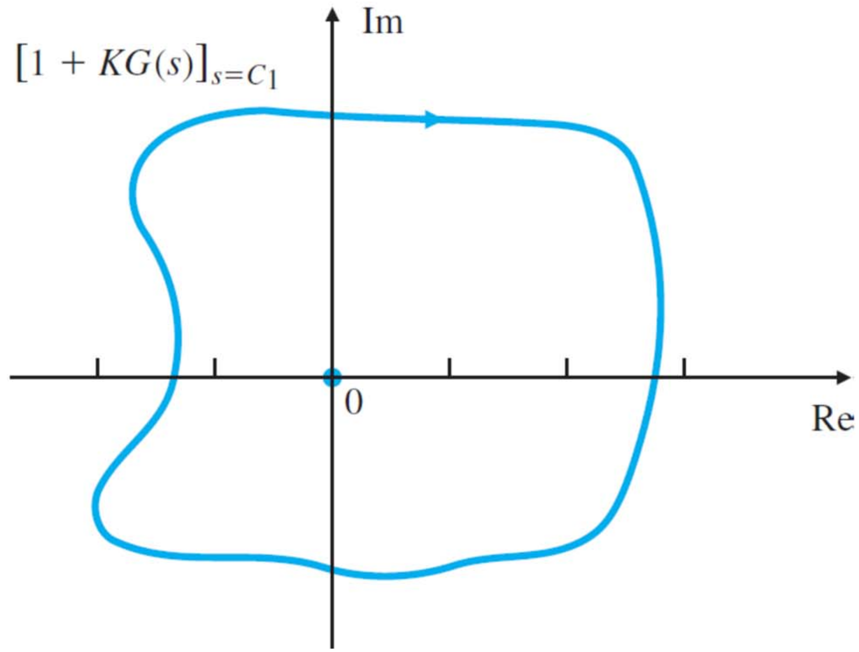

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)}$$

$$1+KG(s)=0, 1+K\frac{b(s)}{a(s)}=0, \frac{a(s)+Kb(s)}{a(s)}=0$$

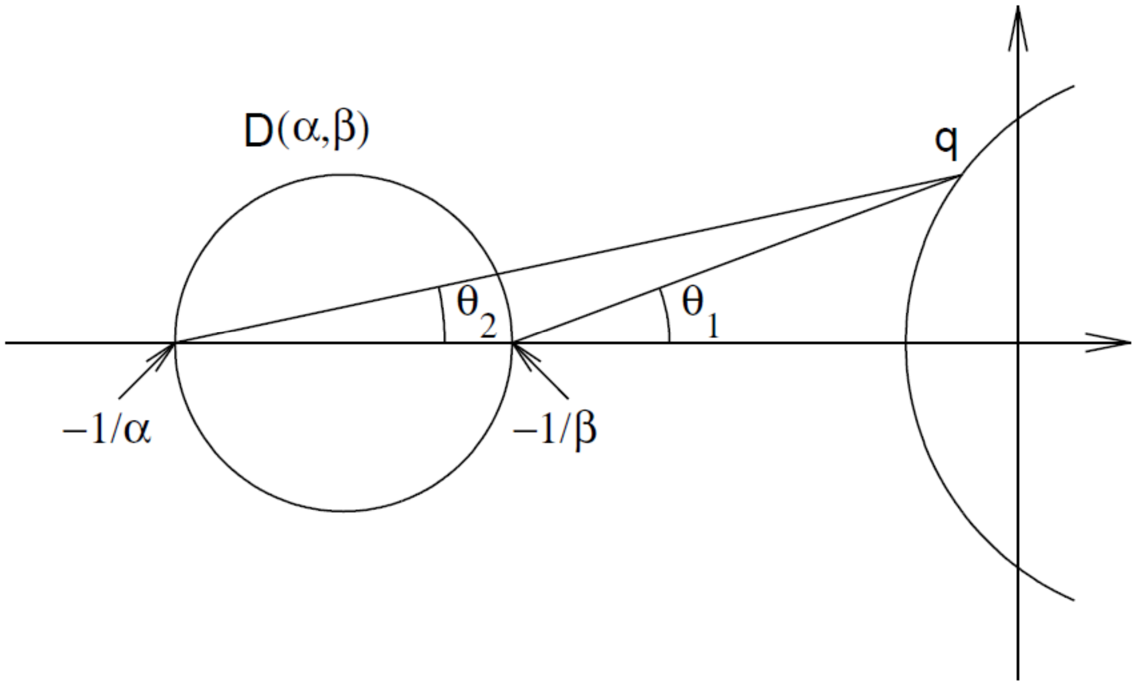
$$N = Z - P \Rightarrow Z = 0, N = -P$$

# Nyquist Criterion



$$\mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] > 0, \ \ \forall\, \omega \in [0, \infty]$$




The system is absolutely stable if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha,\beta)$ and encircles it $m$ times in the counterclockwise direction

Case 2: $\alpha = 0$

$$1 + \beta G(s)$$

$$\mathrm{Re}[1 + \beta G(j\omega)] > 0, \;\; \forall\, \omega \in [0, \infty]$$

$$\mathrm{Re}[G(j\omega)] > -\frac{1}{\beta}, \;\; \forall\, \omega \in [0, \infty]$$

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\mathrm{Re}[s] = -1/\beta$

Case 3: $\alpha < 0 < \beta$

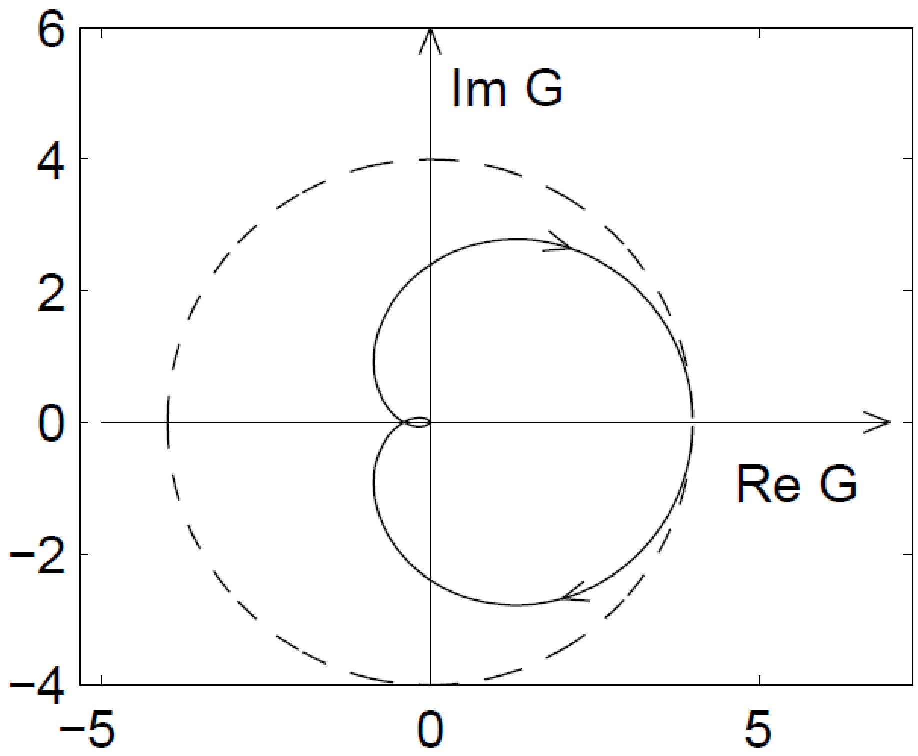
$$\mathrm{Re}\left[\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right] > 0 \quad \Leftrightarrow \quad \mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] < 0$$

The Nyquist plot of $G(j\omega)$ must lie inside the disk $D(\alpha, \beta)$.
The Nyquist plot cannot encircle the point $-(1/\alpha) + j0$.
From the Nyquist criterion, $G(s)$ must be Hurwitz

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies in the interior of the disk $D(\alpha, \beta)$

# Example

$$G(s) = \frac{4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$



Apply Case 3 with center $(0, 0)$ and radius $= 4$

Sector is $(-0.25, 0.25)$

Apply Case 3 with center $(1.5, 0)$ and radius $= 2.834$

Sector is $[-0.227, 0.714]$

Apply Case 2

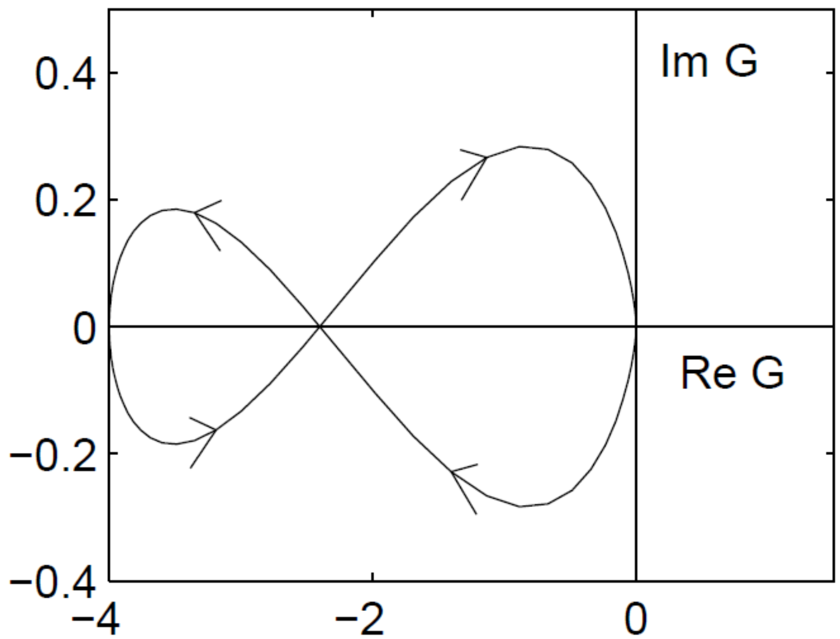
The Nyquist plot is to the right of $\mathrm{Re}[s] = -0.857$

Sector is $[0, 1.166]$

$[0, 1.166]$ includes the saturation nonlinearity

## Example

$$G(s) = \frac{4}{(s-1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

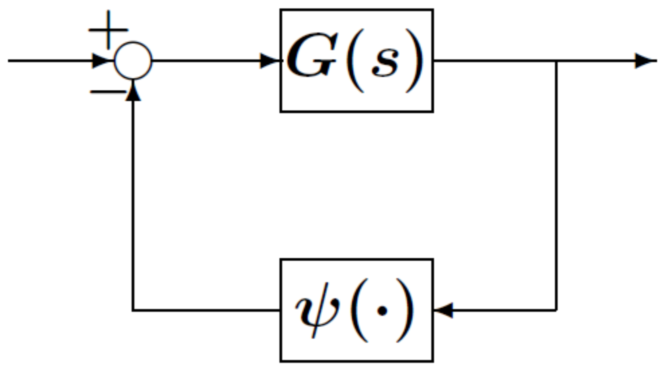



$G$ is not Hurwitz

Apply Case 1

Center $= (-3.2, 0)$, Radius $= 0.168 \Rightarrow [0.2969, 0.3298]$

# Popov Criterion

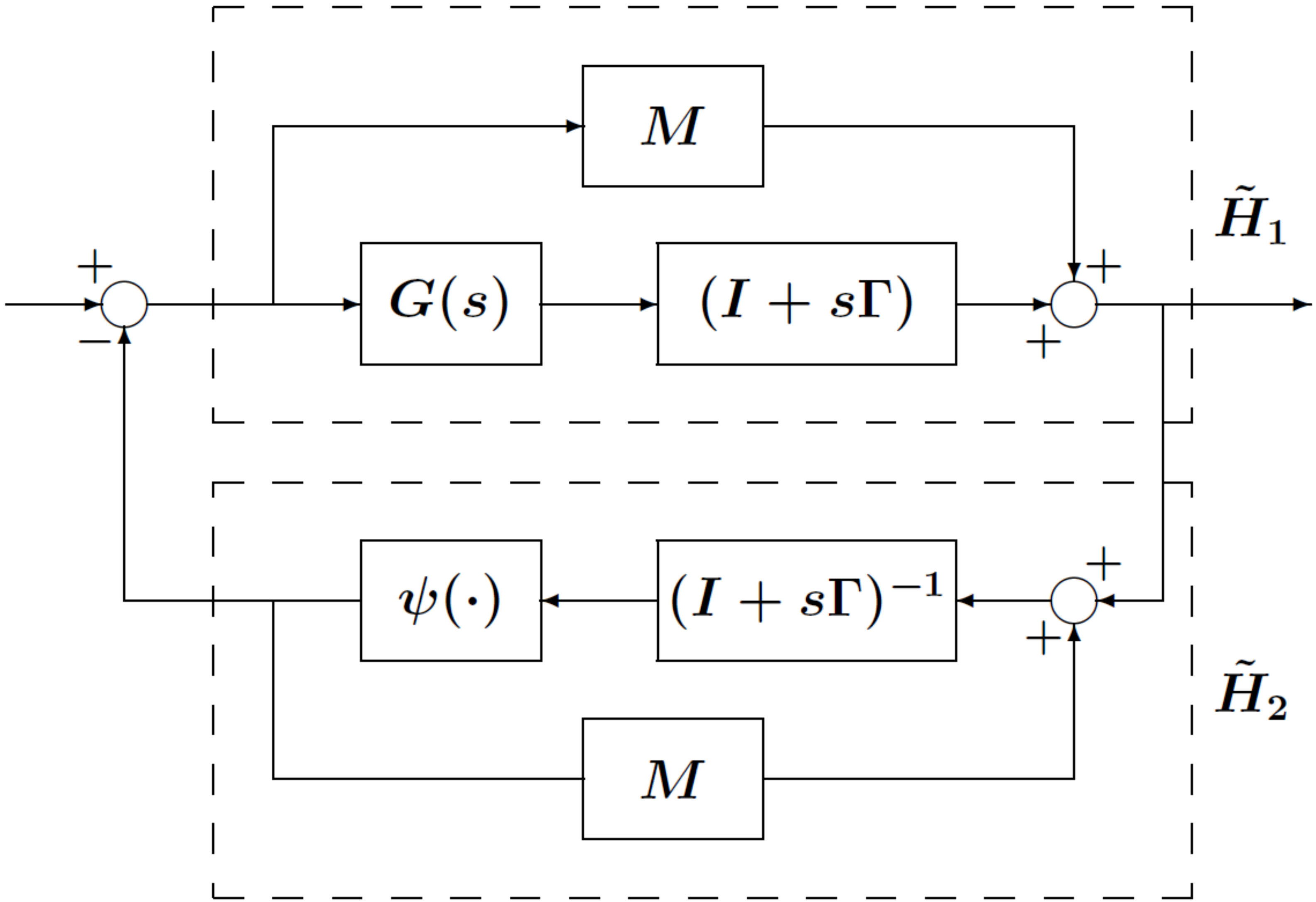



$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ u_i &= -\psi_i(y_i), \;\; 1 \le i \le p \end{aligned}$$

$$\psi_i \in [0, k_i], \quad 1 \le i \le p, \quad (0 < k_i \le \infty)$$

$$G(s) = C(sI - A)^{-1}B$$

$$\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_p), \quad M = \mathrm{diag}(1/k_1, \cdots, 1/k_p)$$




Show that $\tilde{H}_1$ and $\tilde{H}_2$ are passive

$$
\begin{aligned}
& M + (I + s\Gamma)G(s) \\
& \quad = M + (I + s\Gamma)C(sI - A)^{-1}B \\
& \quad = M + C(sI - A)^{-1}B + \Gamma C s(sI - A)^{-1}B \\
& \quad = M + C(sI - A)^{-1}B + \Gamma C(sI - A + A)(sI - A)^{-1}B \\
& \quad = (C + \Gamma C A)(sI - A)^{-1}B + M + \Gamma C B
\end{aligned}
$$

If $M + (I + s\Gamma)G(s)$ is SPR, then $\tilde{H}_1$ is strictly passive with the storage function $V_1 = \frac{1}{2}x^T P x$, where $P$ is given by the KYP equations

$$
\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= (C + \Gamma C A)^T - L^T W \\
W^T W &= 2M + \Gamma C B + B^T C^T \Gamma
\end{aligned}
$$

$\tilde{H}_2$ consists of $p$ decoupled components:

$$\gamma_i \dot{z}_i = -z_i + \frac{1}{k_i}\psi_i(z_i) + \tilde{e}_{2i}, \quad \tilde{y}_{2i} = \psi_i(z_i)$$

$$V_{2i} = \gamma_i \int_0^{z_i} \psi_i(\sigma)\; d\sigma$$

$$\begin{aligned} \dot{V}_{2i} &= \gamma_i \psi_i(z_i)\dot{z}_i = \psi_i(z_i)\left[-z_i + \tfrac{1}{k_i}\psi_i(z_i) + \tilde{e}_{2i}\right] \\ &= y_{2i}e_{2i} + \tfrac{1}{k_i}\psi_i(z_i)\left[\psi_i(z_i) - k_i z_i\right] \end{aligned}$$

$$\psi_i \in [0, k_i] \;\Rightarrow\; \psi_i(\psi_i - k_i z_i) \geq 0 \;\Rightarrow\; \dot{V}_{2i} \leq y_{2i}e_{2i}$$

$\tilde{H}_2$ is passive with the storage function
$V_2 = \sum_{i=1}^{p} \gamma_i \int_0^{z_i} \psi_i(\sigma)\; d\sigma$

Use $$V = \frac{1}{2}x^T P x + \sum_{i=1}^{p} \gamma_i \int_0^{y_i} \psi_i(\sigma)\, d\sigma$$

as a Lyapunov function candidate for the original feedback connection

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad u = -\psi(y)$$

$$\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^T P \dot{x} + \tfrac{1}{2}\dot{x}^T P x + \psi^T(y)\Gamma \dot{y} \\
&= \tfrac{1}{2}x^T (PA + A^T P) x + x^T P B u \\
&\quad + \psi^T(y)\Gamma C(Ax + Bu) \\
&= -\tfrac{1}{2}x^T L^T L x - \tfrac{1}{2}\varepsilon x^T P x \\
&\quad + x^T (C^T + A^T C^T \Gamma - L^T W) u \\
&\quad + \psi^T(y)\Gamma C A x + \psi^T(y)\Gamma C B u
\end{aligned}$$

$$\begin{aligned}\dot{V} &= -\tfrac{1}{2}\varepsilon x^T P x - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) \\ &\quad - \psi(y)^T[y - M\psi(y)] \\ &\le -\tfrac{1}{2}\varepsilon x^T P x - \psi(y)^T[y - M\psi(y)]\end{aligned}$$

$$\psi_i \in [0, k_i] \;\Rightarrow\; \psi(y)^T[y - M\psi(y)] \ge 0 \;\Rightarrow\; \dot{V} \le -\tfrac{1}{2}\varepsilon x^T P x$$

The origin is globally asymptotically stable

Popov Criterion: The system is absolutely stable if, for $1 \le i \le p$, $\psi_i \in [0, k_i]$ and there exists a constant $\gamma_i \ge 0$, with $(1 + \lambda_k \gamma_i) \ne 0$ for every eigenvalue $\lambda_k$ of $A$, such that $M + (I + s\Gamma)G(s)$ is strictly positive real

## Scalar case

$$\frac{1}{k} + (1 + s\gamma)G(s)$$

is SPR if $G(s)$ is Hurwitz and

$$\frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega\text{Im}[G(j\omega)] > 0, \quad \forall\, \omega \in [0, \infty)$$

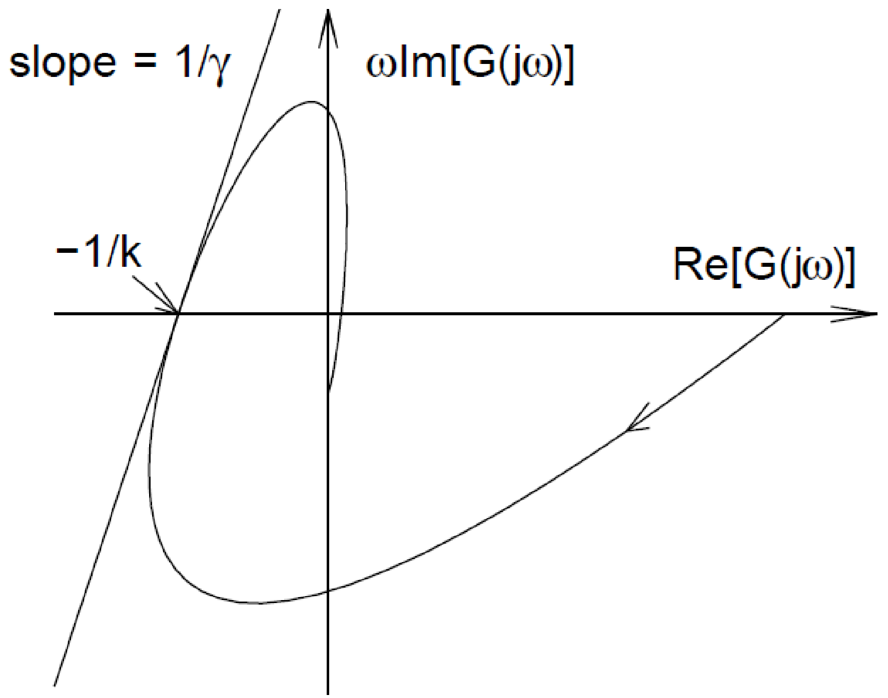
If

$$\lim_{\omega\to\infty} \left\{ \frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega\text{Im}[G(j\omega)] \right\} = 0$$

we also need

$$\lim_{\omega\to\infty} \omega^2 \left\{ \frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega\text{Im}[G(j\omega)] \right\} > 0$$

$$\frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega\mathrm{Im}[G(j\omega)] > 0, \quad \forall\, \omega \in [0, \infty)$$




Popov Plot

## Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - h(y), \quad y = x_1$$

$$\dot{x}_2 = -\alpha x_1 - x_2 - h(y) + \alpha x_1, \quad \alpha > 0$$

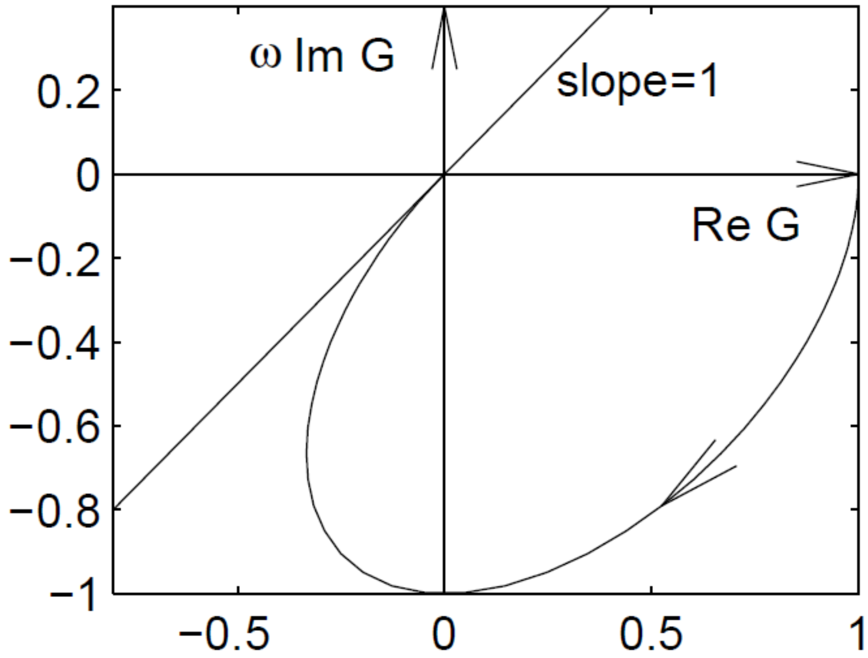
$$G(s) = \frac{1}{s^2 + s + \alpha}, \quad \psi(y) = h(y) - \alpha y$$

$$h \in [\alpha, \beta] \;\Rightarrow\; \psi \in [0, k] \qquad (k = \beta - \alpha > 0)$$

$$\gamma > 1 \;\Rightarrow\; \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \;\; \forall\, \omega \in [0, \infty)$$

and $\displaystyle \lim_{\omega \to \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma\omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0$

The system is absolutely stable for $\psi \in [0, \infty]$ ($h \in [\alpha, \infty]$)




Compare with the circle criterion ($\gamma = 0$)

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \ \ \forall\, \omega \in [0, \infty], \quad \text{for } k < 1 + 2\sqrt{\alpha}$$