Chapter 7
Frequency Domain Analysis of Feedback Systems

Describing Function

Frequency Response Method

- Describing a linear system by a complex-valued function, the frequency response
- For nonlinear systems, describing function method can be used to approximately analyze and predict nonlinear behavior
- Hard nonlinearities

An Example of Describing Function Analysis

Consider the Van der Pol equation

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

Let us determine whether there exists a limit cycle in this system and, if so, calculate the amplitude and frequency of the limit cycle (pretending that we have not seen the phase portrait of the Van der Pol equation in chapter 2). To this effect, we first assume the existence of a limit cycle with undetermined amplitude and frequency, and then determine whether the system equation can indeed sustain such a solution. This is

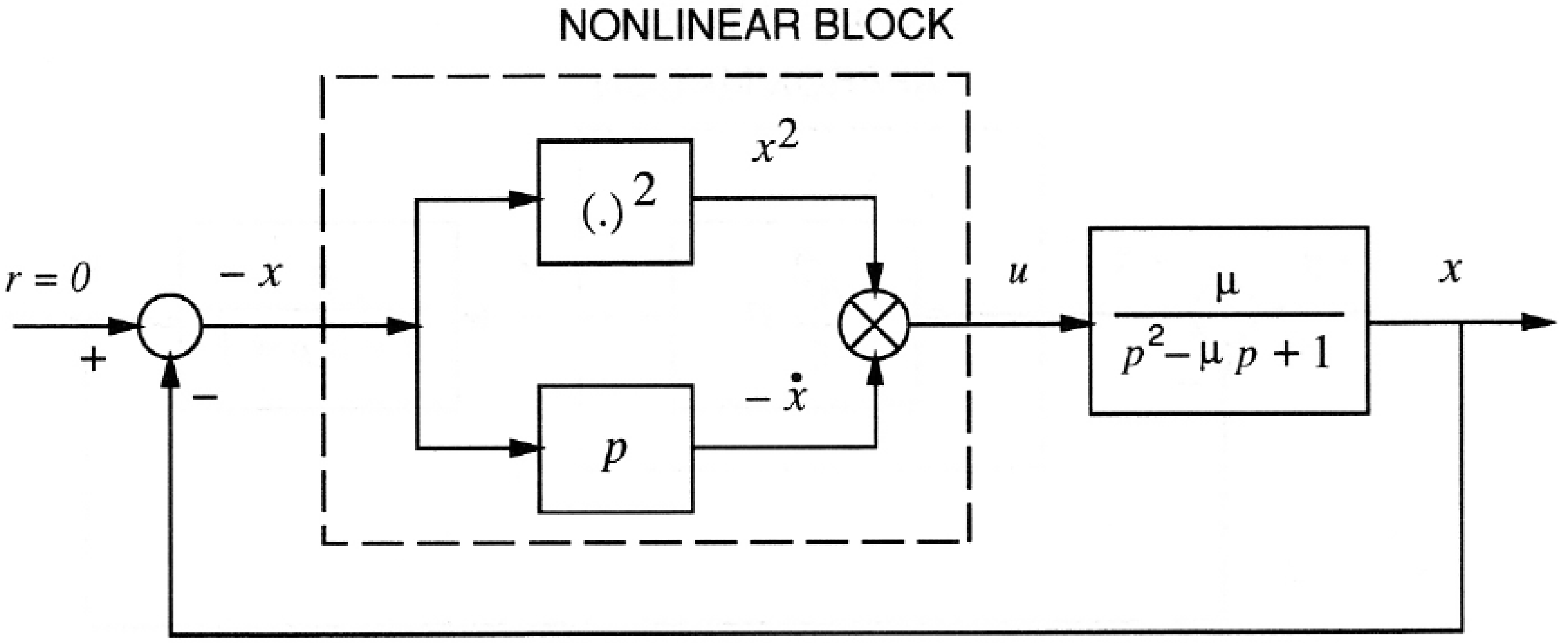
An Example of Describing Function Analysis

$$\ddot{x} - \mu\dot{x} + x + \mu x^2\dot{x} = 0, \frac{\ddot{x} - \mu\dot{x} + x}{\mu} = -x^2\dot{x}$$

$$\frac{\ddot{x} - \mu\dot{x} + x}{\mu} = u, u = -x^2\dot{x}$$

$$p = \frac{d}{dt}, \frac{X(s)}{U(s)} = \frac{\mu}{p^2 - \mu p + 1}$$

An Example of Describing Function Analysis



An Example of Describing Function Analysis

Now let us assume that there is a limit cycle in the system and the oscillation signal x is in the form of

$$x(t) = A\sin(\omega t)$$

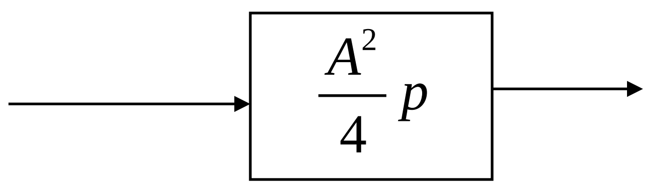
with A being the limit cycle amplitude and ω being the frequency. Thus,

$$\dot{x}(t) = A\omega\cos(\omega t)$$

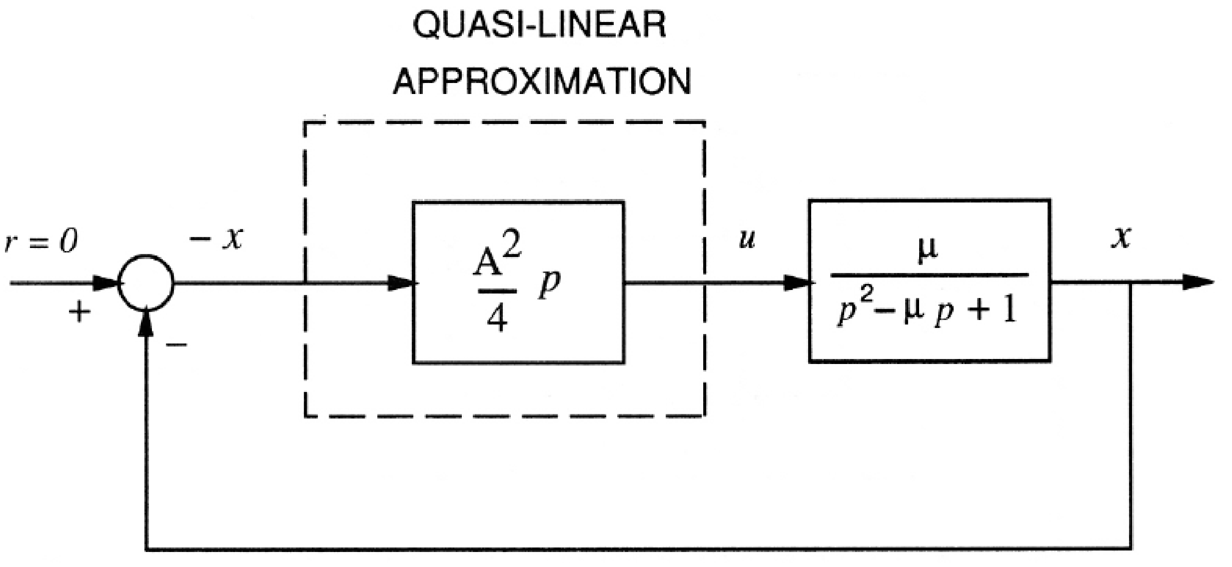
$$u = -x^2\dot{x} = -A^2\sin^2(\omega t)\ A\omega\cos(\omega t)$$

$$= -\frac{A^3\omega}{2}(1-\cos(2\omega t))\cos(\omega t) = -\frac{A^3\omega}{4}(\cos(\omega t) - \cos(3\omega t))$$

$$u \approx -\frac{A^3}{4}\omega\cos\omega t = \frac{A^2}{4}\frac{\mathrm{d}}{\mathrm{d}t}[-A\sin(\omega t)]$$



An Example of Describing Function Analysis



In the frequency domain, this corresponds to

$$u = N(A, \omega)\ (-x)$$

where

$$N(A, \omega) = \frac{A^2}{4}\,(j\omega)$$

An Example of Describing Function Analysis

Since the system is assumed to contain a sinusoidal oscillation, we have

$$x = A\sin(\omega t) = G(j\omega)\,u = G(j\omega)\,N(A,\,\omega)\,(\,-x\,)$$

where $G(j\omega)$ is the linear component transfer function. This implies that

$$1 + \frac{A^2\,(j\omega)}{4}\,\frac{\mu}{(j\omega)^2 - \mu\,(j\omega) + 1} \;=\; 0$$

Solving this equation, we obtain

$$A = 2 \qquad \omega = 1$$

An Example of Describing Function Analysis

Note that in terms of the Laplace variable p, the closed-loop characteristic equation of this system is

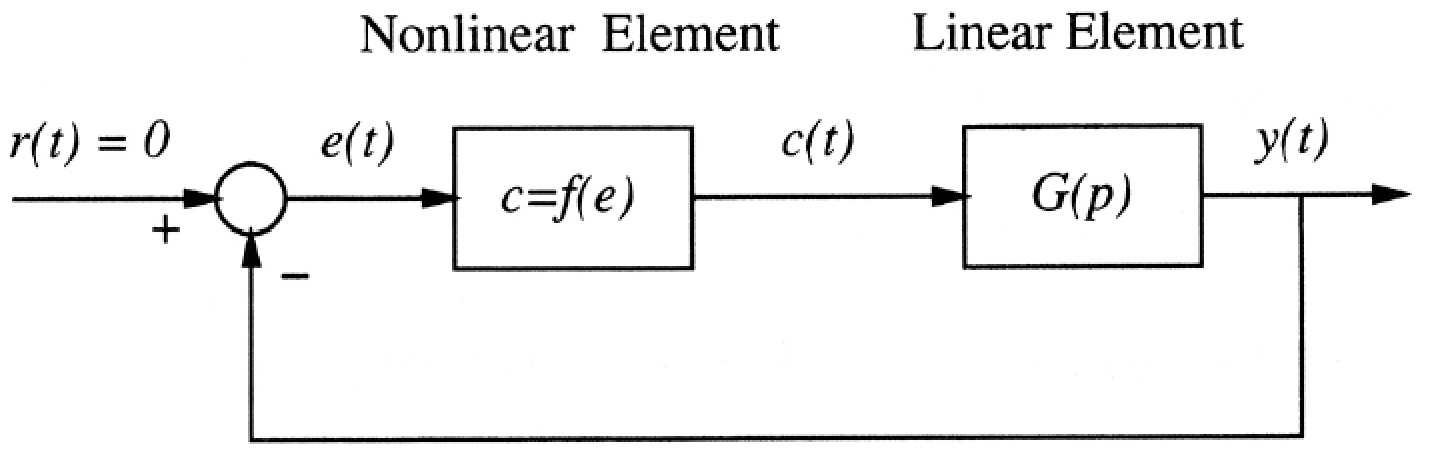
$$1 + \frac{A^2 p}{4} \frac{\mu}{p^2 - \mu p + 1} = 0 \tag{5.3}$$

whose eigenvalues are

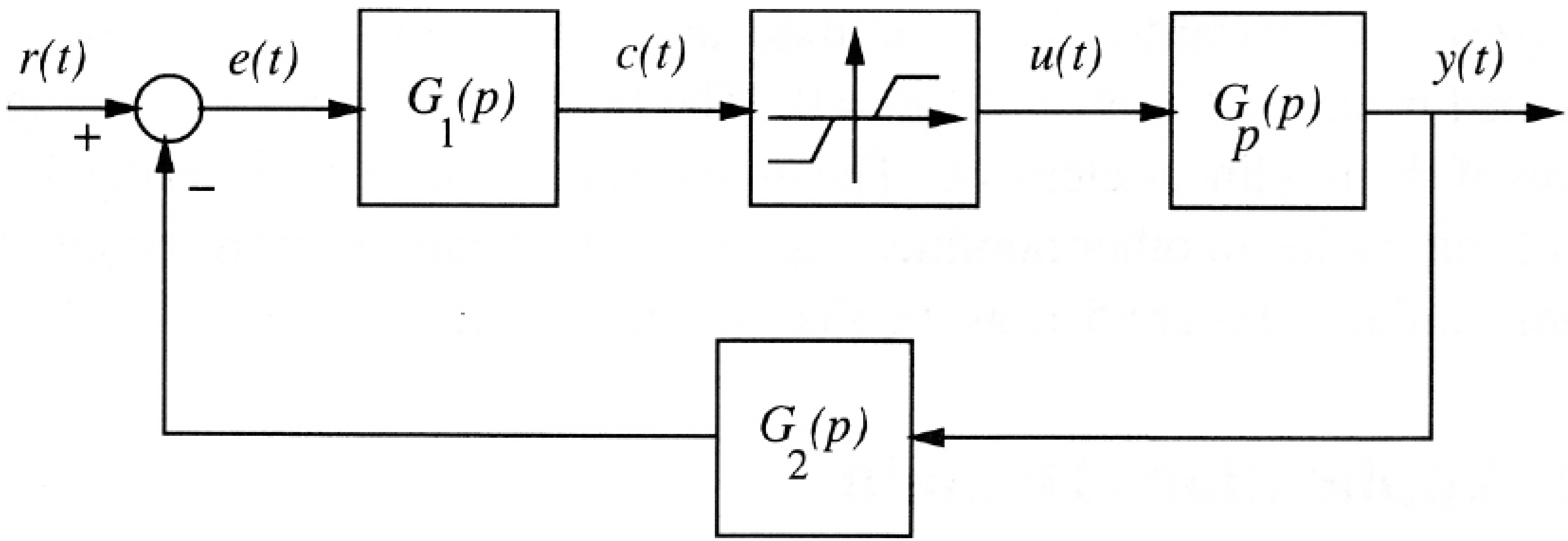
$$\lambda_{1,2} = -\frac{1}{8}\mu(A^2 - 4) \pm \sqrt{\frac{1}{64}\mu^2(A^2-4)^2 - 1} \tag{5.4}$$

Corresponding to $A = 2$, we obtain the eigenvalues $\lambda_{1,2} = \pm j$. This indicates the existence of a limit cycle of amplitude 2 and frequency 1. It is interesting to note neither the amplitude nor the frequency obtained above depends on the parameter μ in Equation 5.1.

Applications Domain



Example 5.1

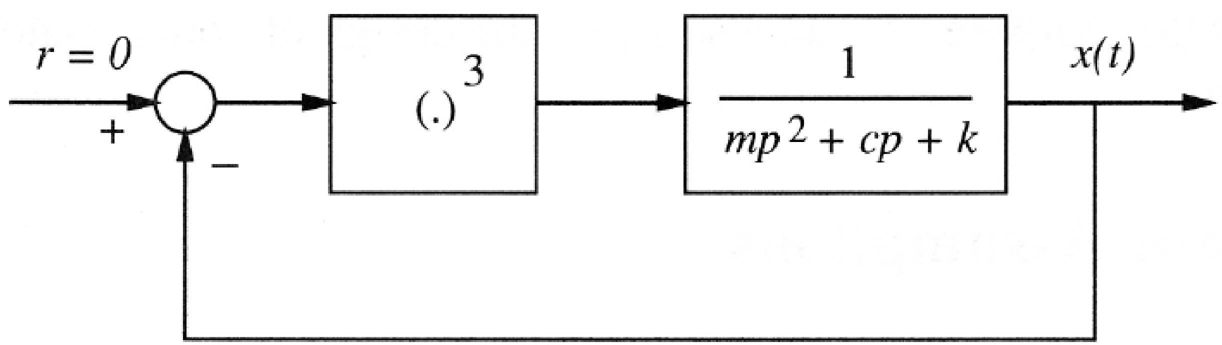


The second class of systems consists of genuinely nonlinear systems whose dynamic equations can actually be rearranged into the form of Figure 5.3. For example, the nonlinear equation

$$m\ddot{x} + c\dot{x} + kx + k_1 x^3 = 0$$

can be rewritten as

$$m\ddot{x} + c\dot{x} + kx = -k_1 x^3$$



Prediction of limit cycles is very important, because limit cycles can occur in any kind of physical nonlinear system. Sometimes, a limit cycle can be desirable. This is the case of limit cycles in the electronic oscillators used in laboratories. Another example is the so-called dither technique which can be used to minimize the negative effects of Coulomb friction in mechanical systems. In most control systems, however, limit cycles are undesirable. This may be due to a number of reasons:

1. limit cycling, as a way of instability, tends to cause poor control accuracy
2. the constant oscillation associated with the limit cycles can cause increasing wear or even mechanical failure of the control system hardware
3. limit cycling may also cause other undesirable effects, such as passenger discomfort in an aircraft under autopilot

Basic Assumptions

1. *there is only a single nonlinear component*
2. *the nonlinear component is time-invariant*
3. *corresponding to a sinusoidal input* $e = \sin(\omega t)$ *, only the fundamental component* $c_1(t)$ *in the output* $c(t)$ *has to be considered*
4. *the nonlinearity is odd*

The third assumption is the *fundamental assumption* of the describing function method. It represents an *approximation*, because the output of a nonlinear element corresponding to a sinusoidal input usually contains higher harmonics besides the fundamental. This assumption implies that the higher-frequency harmonics can all be neglected in the analysis, as compared with the fundamental component. For this assumption to be valid, it is important for the linear element following the nonlinearity to have low-pass properties, *i.e.*,

$$|G(j\omega)| >> |G(jn\omega)| \quad \text{for} \;\; n = 2, 3, \ldots \tag{5.5}$$

This implies that higher harmonics in the output will be filtered out significantly. Thus, the third assumption is often referred to as the *filtering hypothesis*.

Basic Definitions

- Fourier series

$$c(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)] \qquad (5.6)$$

where the Fourier coefficients a_i's and b_i's are generally functions of A and ω, determined by

$$a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t)\, d(\omega t) \qquad (5.7a)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t) \cos(n\omega t)\, d(\omega t) \qquad (5.7b)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} c(t) \sin(n\omega t)\, d(\omega t) \qquad (5.7c)$$

Basic Definitions

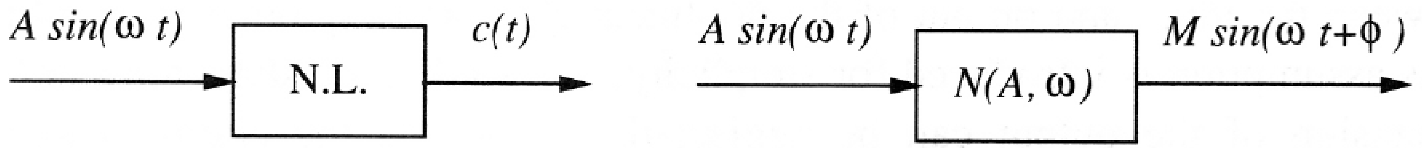


Figure 5.6 : A nonlinear element and its describing function representation

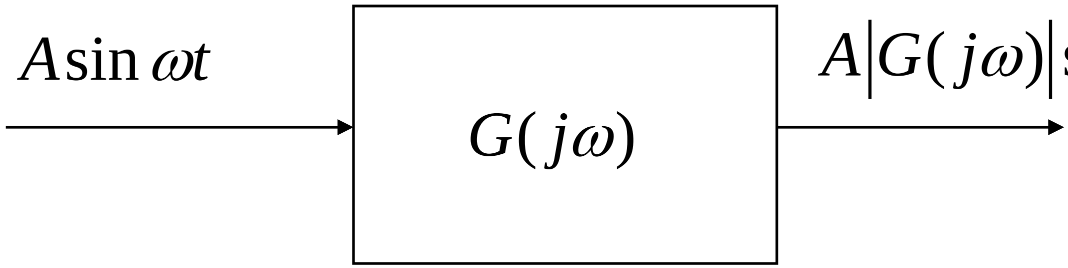
$$c(t) \approx c_1(t) = a_1 \cos(\omega t) + b_1 \sin(\omega t) = M \sin(\omega t + \phi) \tag{5.8}$$

where

$$M(A, \omega) = \sqrt{a_1^2 + b_1^2} \quad \text{and} \quad \phi(A, \omega) = \arctan(a_1 / b_1).$$

Frequency Response

$A\sin\omega t$



$A\left|G(j\omega)\right|\sin(\omega t + \measuredangle G(j\omega))$

$Ae^{j\omega t}$

$A\left|G(j\omega)\right|e^{j\measuredangle G(j\omega)}e^{j\omega t}$

$$G(j\omega) = \frac{A\left|G(j\omega)\right|e^{j\measuredangle G(j\omega)}e^{j\omega t}}{Ae^{j\omega t}}$$

Definition of Describing Function

$$c_1 = M e^{j(\omega t + \phi)} = (b_1 + j a_1)\, e^{j\omega t}$$

the *describing function* of the nonlinear element to be *the complex ratio of the fundamental component of the nonlinear element by the input sinusoid*, i.e.,

$$N(A, \omega) = \frac{M e^{j(\omega t + \phi_1)}}{A e^{j\omega t}} = \frac{M}{A} e^{j\phi} = \frac{1}{A}(b_1 + j a_1) \qquad (5.9)$$

Computing Describing Function

Example 5.2: Describing function of a hardening spring

The characteristics of a hardening spring are given by

$$y = x + x^3/2$$

with x being the input and y being the output. Given an input $x(t) = A\sin(\omega t)$, the output $y(t) = A\sin(\omega t) + A^3 \sin^3(\omega t)/2$ can be expanded as a Fourier series, with the fundamental being

$$y_1(t) = a_1 \cos\omega t + b_1 \sin\omega t$$

Computing Describing Function

Because $y(t)$ is an odd function, one has $a_1 = 0$, according to (5.7). The coefficient b_1 is

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} [A\sin(\omega t) + A^3 \sin^3(\omega t)/2]\sin(\omega t)\, d(\omega t) = A + \frac{3}{8}A^3$$

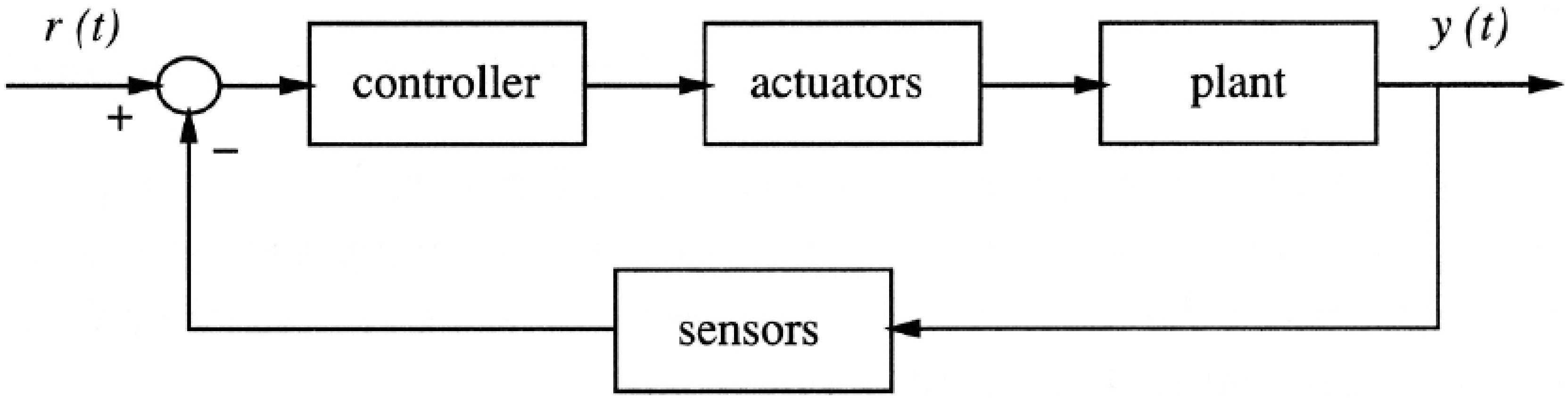
Therefore, the fundamental is

$$y_1 = (A + \frac{3}{8}A^3)\sin(\omega t)$$

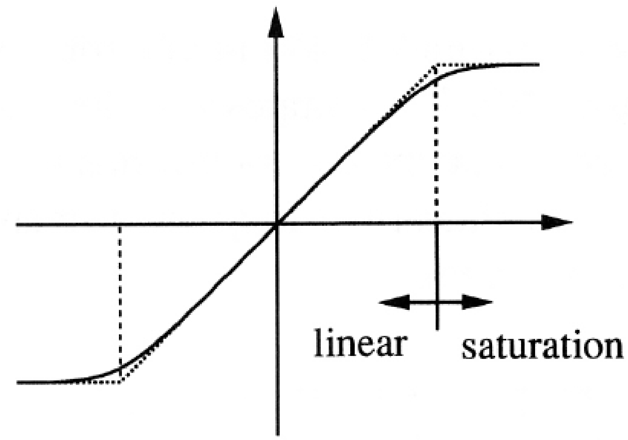
and the describing function of this nonlinear component is

$$N(A,\omega) = N(A) = 1 + \frac{3}{8}A^2$$

Common Nonlinearities in Control Systems

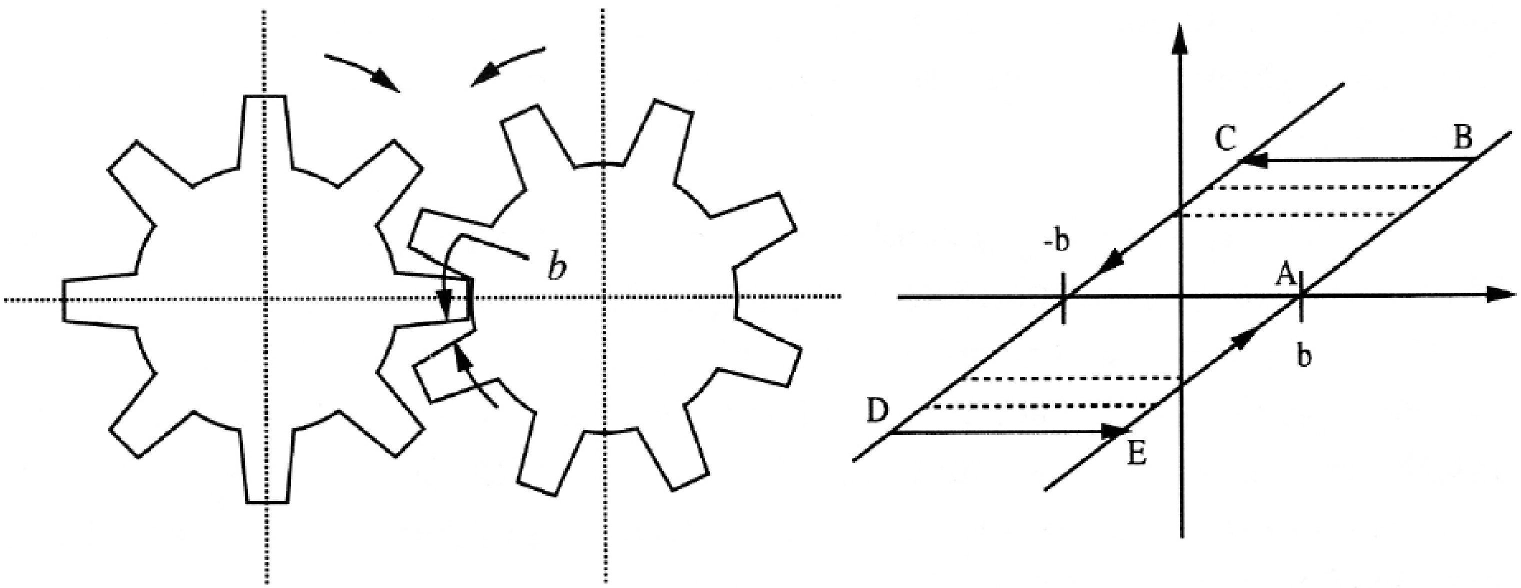


- Saturation
- On-off nonlinearity
- Dead-zone

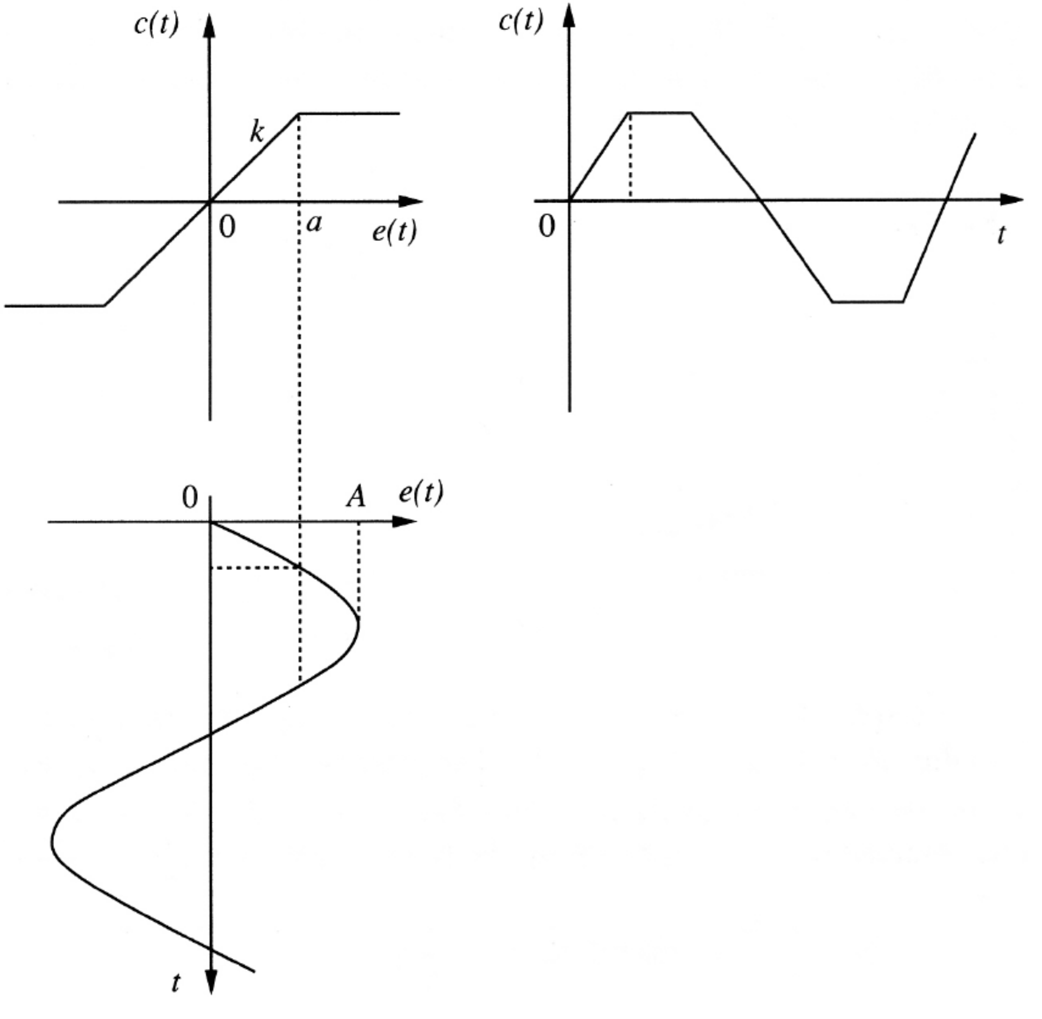


Common Nonlinearities in Control Systems

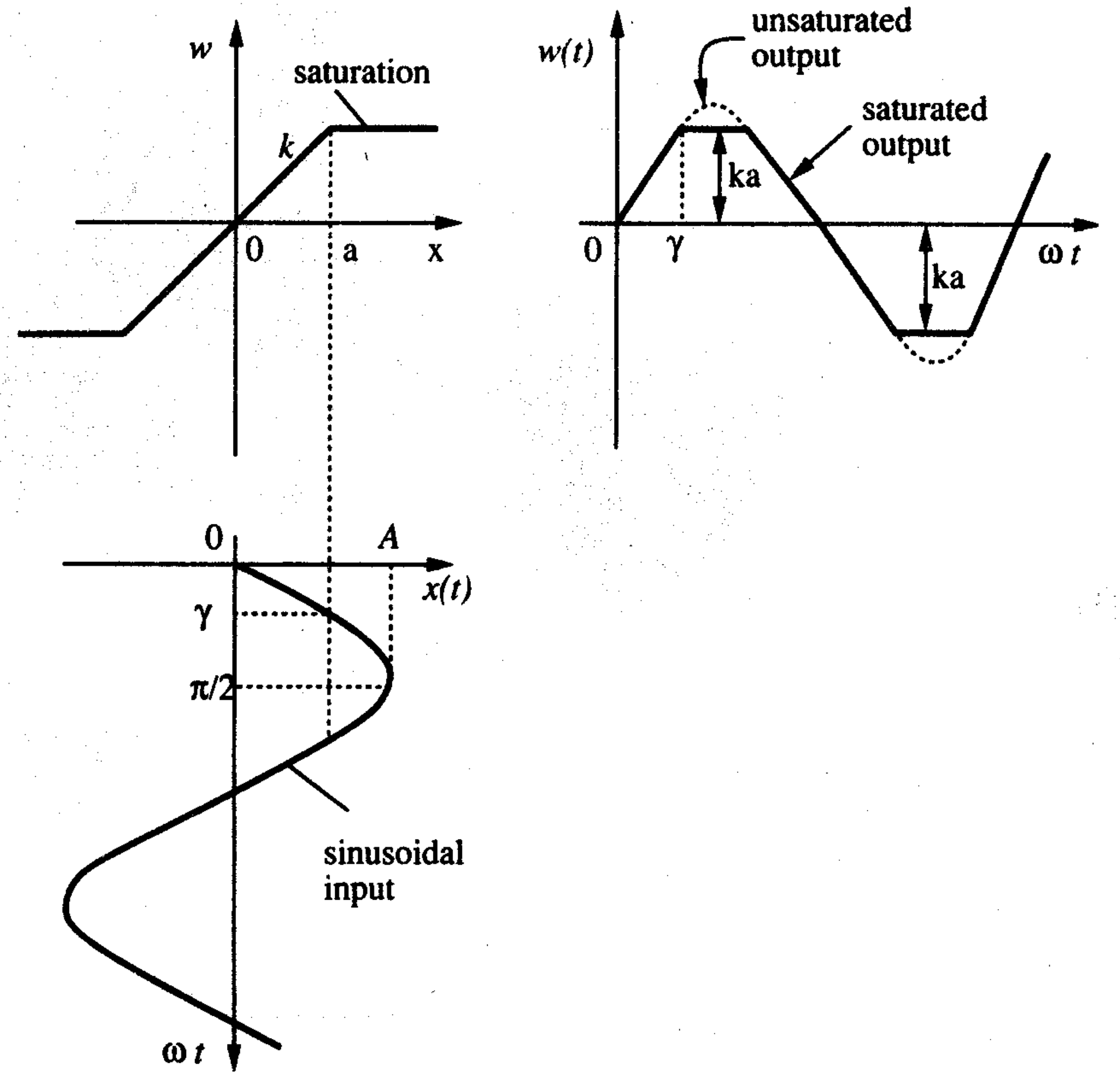
- Backlash and hysteresis



Describing Function of Saturation Nonlinearity



Describing Function of Saturation Nonlinearity



Describing Function of Saturation Nonlinearity

Consider the input $e(t) = A\sin(\omega t)$. If $A \le a$, then the input remains in the linear range, and therefore, the output is $y(t) = kA\sin(\omega t)$. Hence, the describing function is simply a constant k.

Now consider the case $A > a$. The input and the output are plotted in Figure 5.10. The output can be expressed as

$$c(t) = \begin{cases} kA\sin(\omega t) & 0 \le \omega t \le \omega t_1 \\ ka & \omega t_1 < \omega t \le \pi/2 \end{cases}$$

where $\omega t_1 = \sin^{-1}(a/A)$. The odd nature of $c(t)$ implies that $a_1 = 0$ and the symmetry over the four quarters of a period implies that

Describing Function of Saturation Nonlinearity

$$b_1 = \frac{4}{\pi}\int_o^{\pi/2} c(t)\sin(\omega t)\, d(\omega t)$$

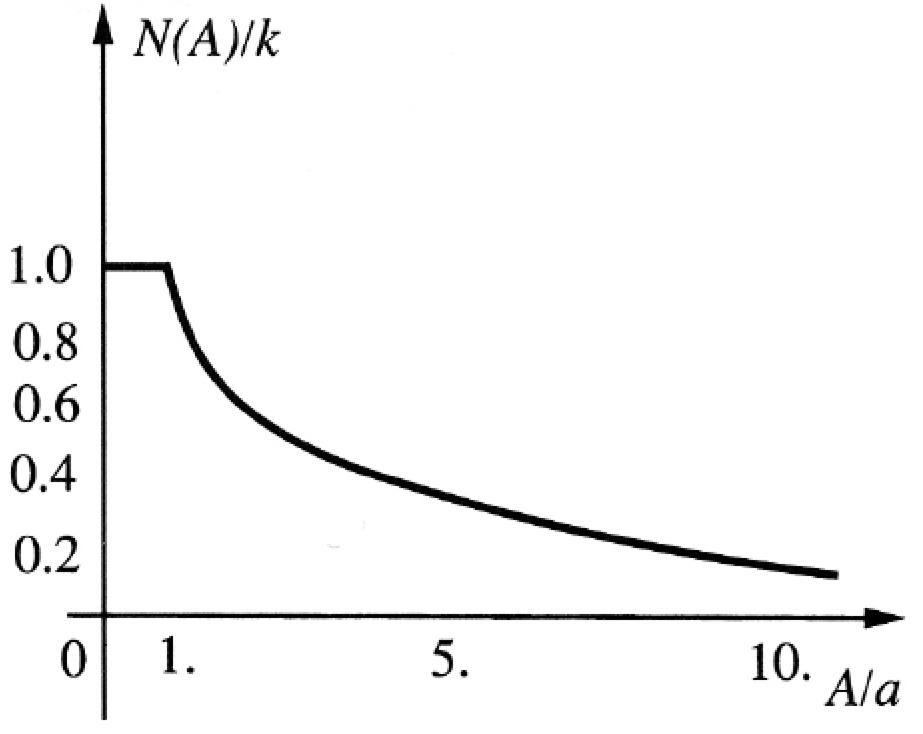
$$= \frac{4}{\pi}\int_o^{\omega t_1} kA\sin^2(\omega t)\, d(\omega t) + \frac{4}{\pi}\int_{\omega t_1}^{\pi/2} ka\sin(\omega t)\, d(\omega t)$$

$$= \frac{2kA}{\pi}[\ \omega t_1 + \frac{a}{A}\sqrt{1-\frac{a^2}{A^2}}\ ] \qquad (5.10)$$

Therefore, the describing function is

$$N(A) = \frac{b_1}{A} = \frac{2k}{\pi}[\sin^{-1}\frac{a}{A} + \frac{a}{A}\sqrt{1-\frac{a^2}{A^2}}\ ] \qquad (5.11)$$

Describing Function of Saturation Nonlinearity



Describing Function of Relay (on-off) Nonlinearity

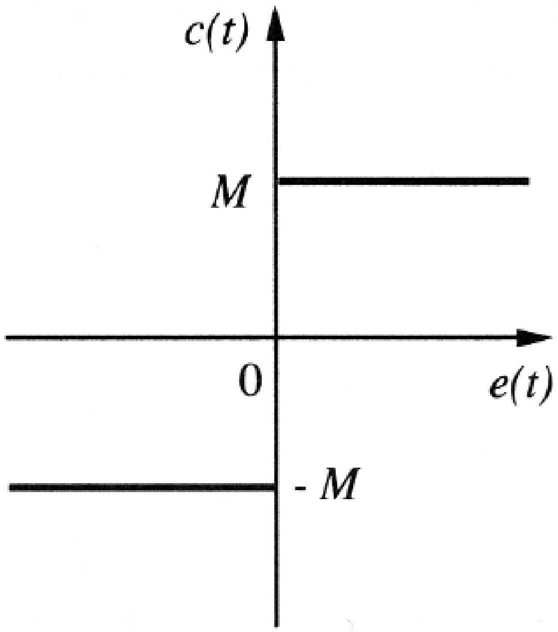
As a special case, one can obtain the describing function for the relay-type (on-off) nonlinearity shown in Figure 5.12. This case corresponds to shrinking the linearity range in the saturation function to zero, *i.e.*, $a \to 0$, $k \to \infty$, but $ka = M$. Though b_1 can be obtained from (5.10) by taking the limit, it is more easily obtained directly as

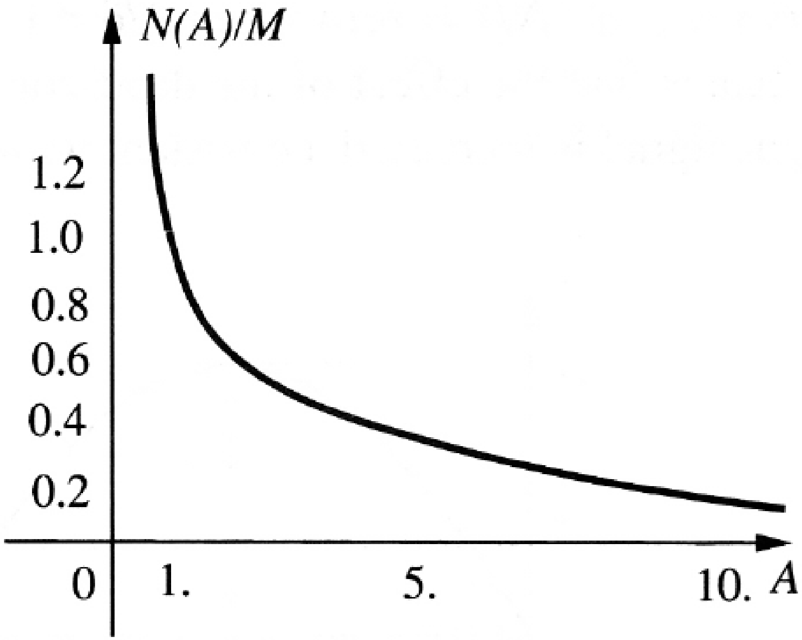
$$b_1 = \frac{4}{\pi} \int_0^{\pi/2} M \sin(\omega t)\, d(\omega t) = \frac{4}{\pi} M$$

Therefore, the describing function of the relay nonlinearity is

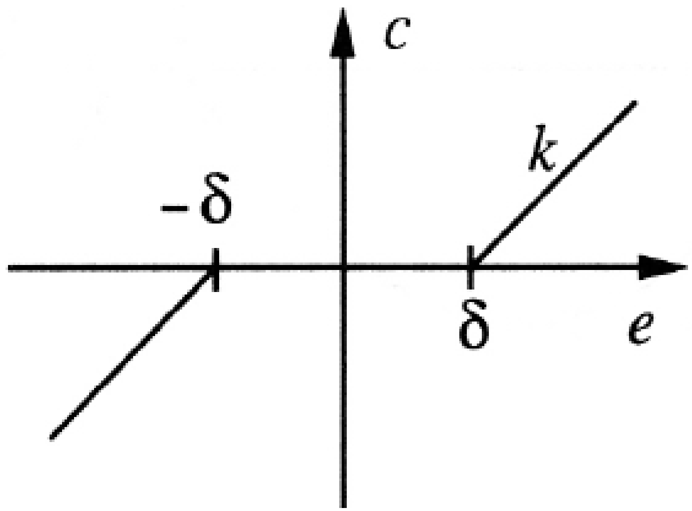
$$N(A) = \frac{4M}{\pi A} \qquad (5.12)$$

Describing Function of Relay (on-off) Nonlinearity

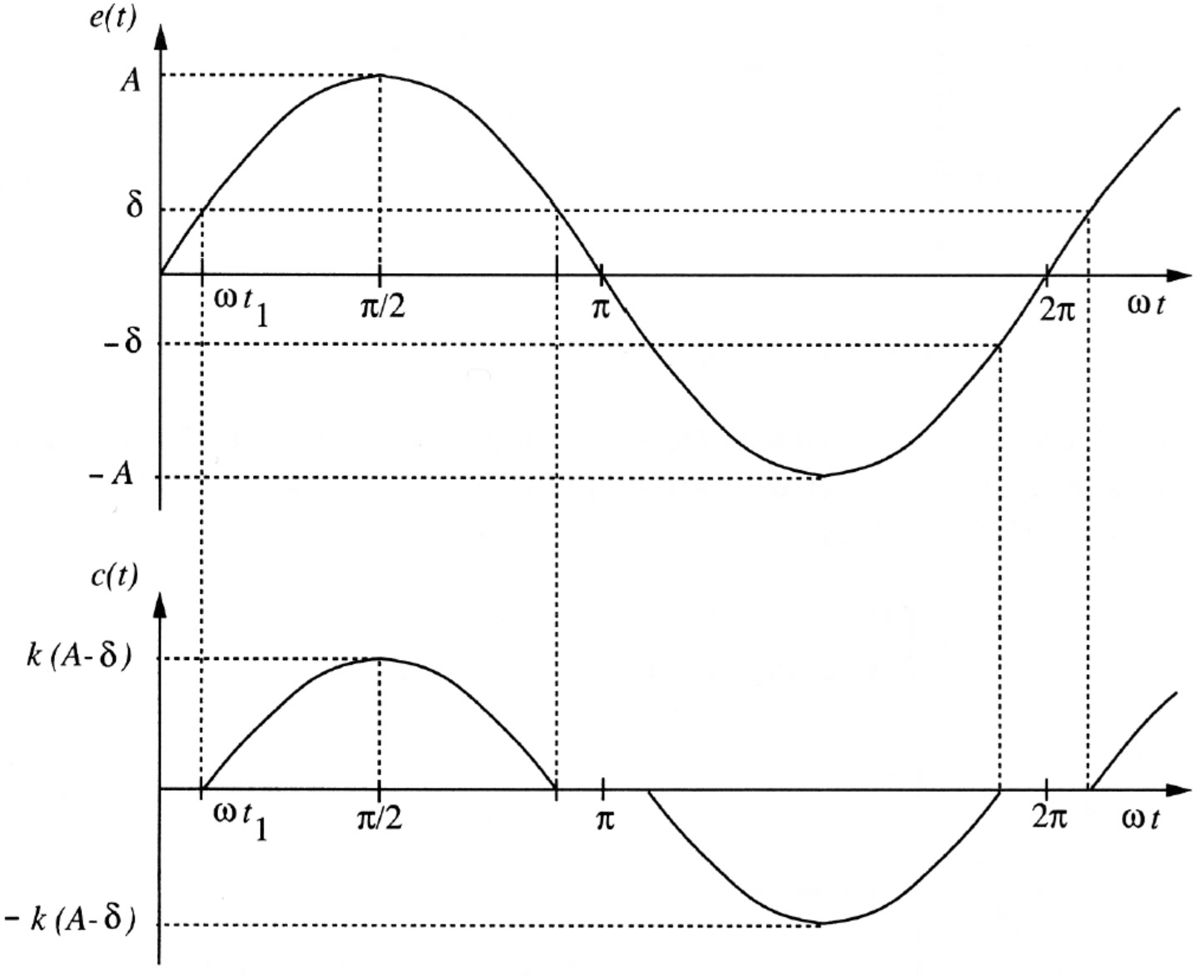




Describing Function of Dead-zone Nonlinearity



Describing Function of Dead-zone Nonlinearity



Describing Function of Dead-zone Nonlinearity

$$c(t) = \begin{cases} 0 & 0 \le \omega t \le \omega t_1 \\ k(A\sin(\omega t) - \delta) & \omega t_1 \le \omega t \le \pi/2 \end{cases}$$

where $\omega t_1 = \sin^{-1}(\delta/A)$. The coefficient b_1 can be computed as follows

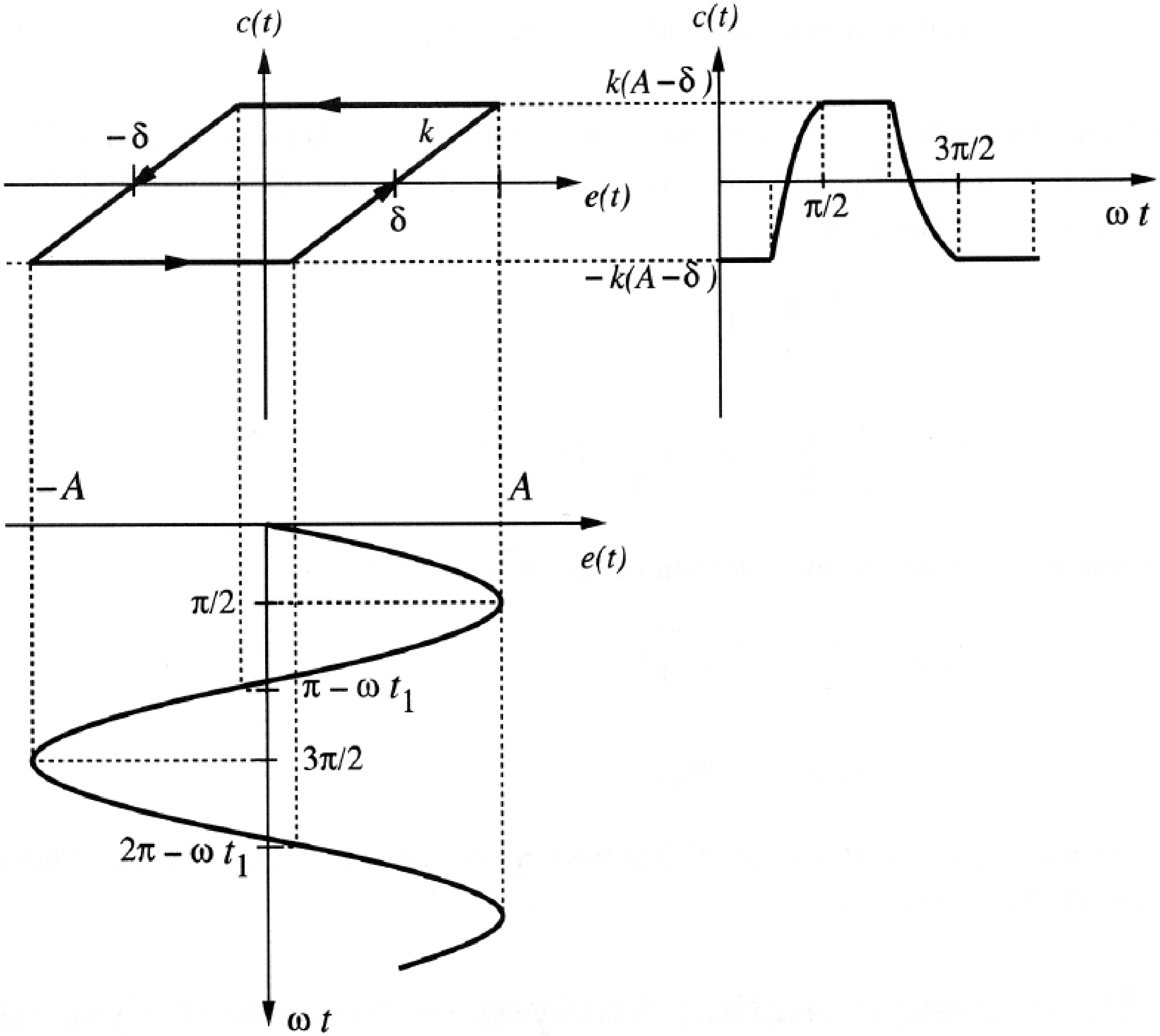
$$b_1 = \frac{4}{\pi}\int_0^{\pi/2} c(t)\sin(\omega t)\, d(\omega t) = \frac{4}{\pi}\int_{\omega t_1}^{\pi/2} k(A\sin(\omega t) - \delta)\sin(\omega t)\, d(\omega t)$$

$$= \frac{2kA}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\frac{\delta}{A} - \frac{\delta}{A}\sqrt{1 - \frac{\delta^2}{A^2}}\right) \qquad (5.13)$$

Therefore, the describing function is

$$N(A) = \frac{2k}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\frac{\delta}{A} - \frac{\delta}{A}\sqrt{1 - \frac{\delta^2}{A^2}}\right)$$

Describing Function of Backlash Nonlinearity



Describing Function of Backlash Nonlinearity

$$c(t) = (A - \delta)k \qquad \frac{\pi}{2} < \omega t \le \pi - \omega t_1$$

$$c(t) = (A\sin(\omega t) + \delta)k \qquad \pi - \omega t_1 < \omega t \le \frac{3\pi}{2}$$

$$c(t) = -(A - \delta)k \qquad \frac{3\pi}{2} < \omega t \le 2\pi - \omega t_1$$

$$c(t) = (A\sin(\omega t) - \delta)k \qquad 2\pi - \omega t_1 < \omega t \le \frac{5\pi}{2}$$

$$a_1 = \frac{4k\delta}{\pi}\left(\frac{\delta}{A} - 1\right)$$

$$b_1 = \frac{Ak}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{2\delta}{A} - 1\right) - \left(\frac{2\delta}{A} - 1\right)\sqrt{1 - \left(\frac{2\delta}{A} - 1\right)^2}\right.$$

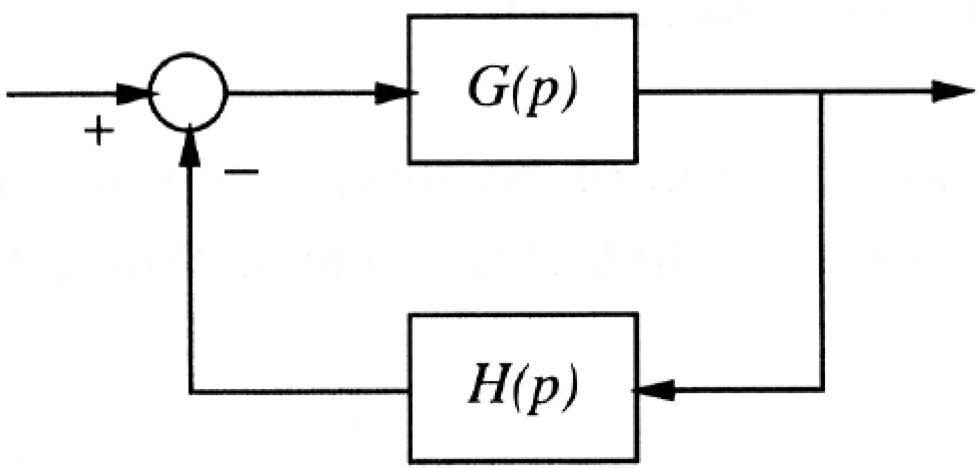
$$|N(A)| = \frac{1}{A}\sqrt{a_1^2 + b_1^2}$$

$$\underline{/N(A)} = \tan^{-1}(a_1/b_1)$$

The Nyquist Criterion and Its Extension

$$\delta(p) = 1 + G(p)\,H(p) = 0$$

$$G(p)\,H(p) = -1$$

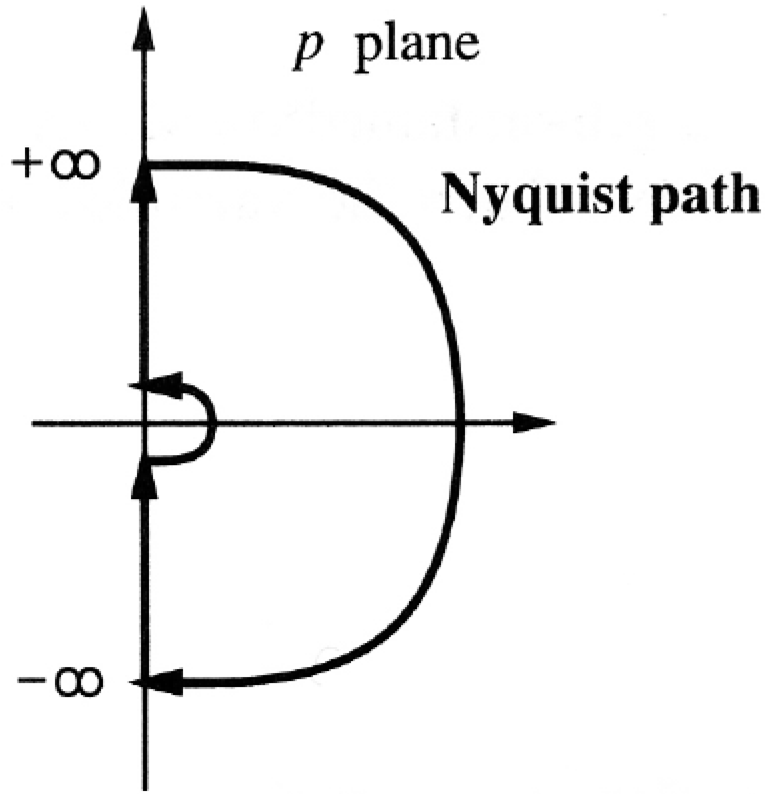


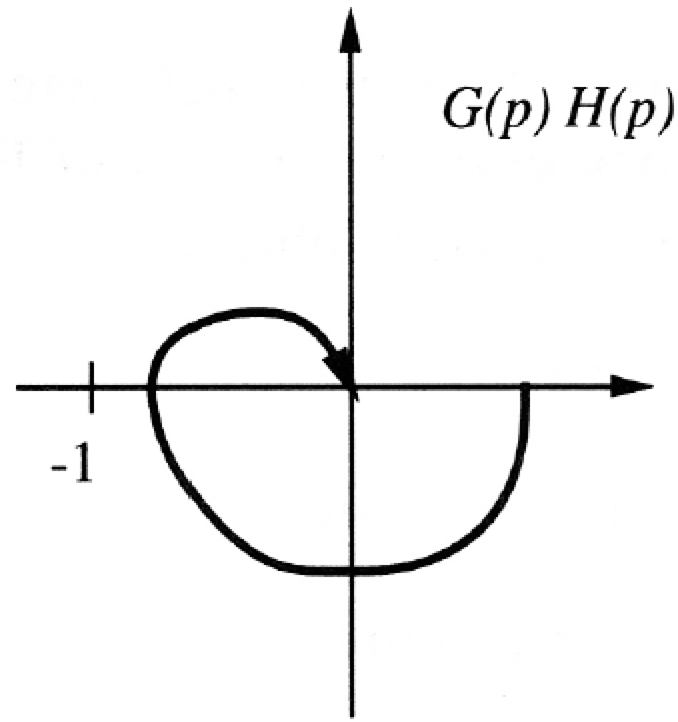
The Nyquist Criterion and Its Extension

1. draw, in the p plane, a so-called Nyquist path enclosing the right-half plane
2. map this path into another complex plane through $G(p)H(p)$
3. determine N, the number of clockwise encirclements of the plot of $G(p)H(p)$ around the point $(-1,0)$
4. compute Z, the number of zeros of the loop transfer function $\delta(p)$ in the right-half p plane, by

$$Z = N + P \quad , \text{ where } P \text{ is the number of unstable poles of } \delta(p)$$

The Nyquist Criterion and Its Extension

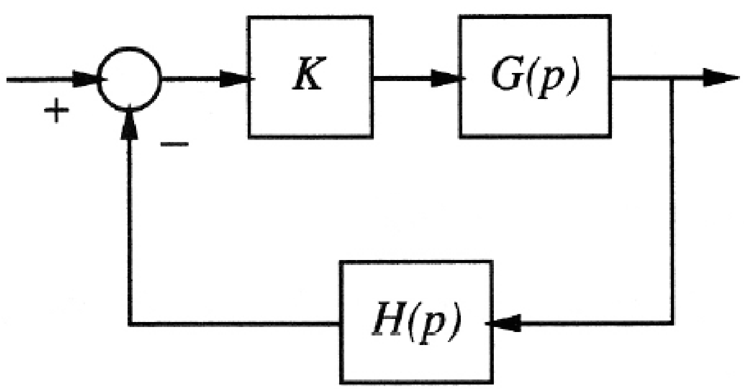


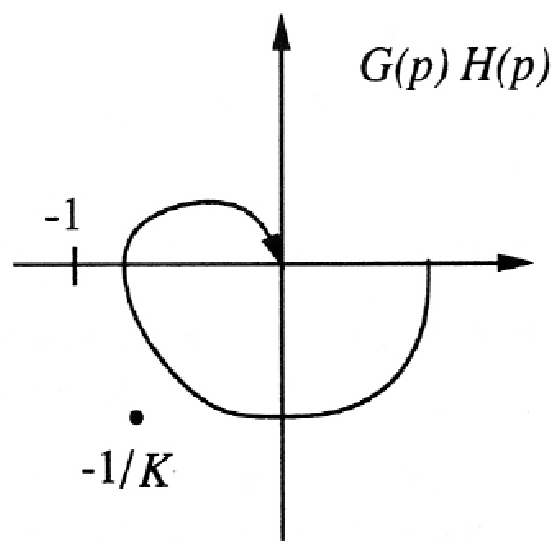


The Nyquist Criterion and Its Extension

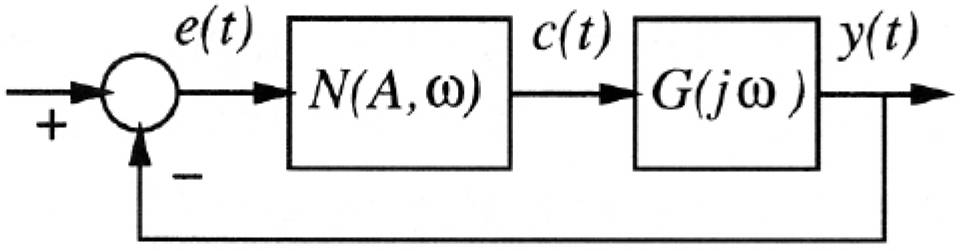
$$\delta(p) = 1 + K\,G(p)H(p)$$

$$G(p)\,H(p) = -1/K$$





Existence of Limit Cycles

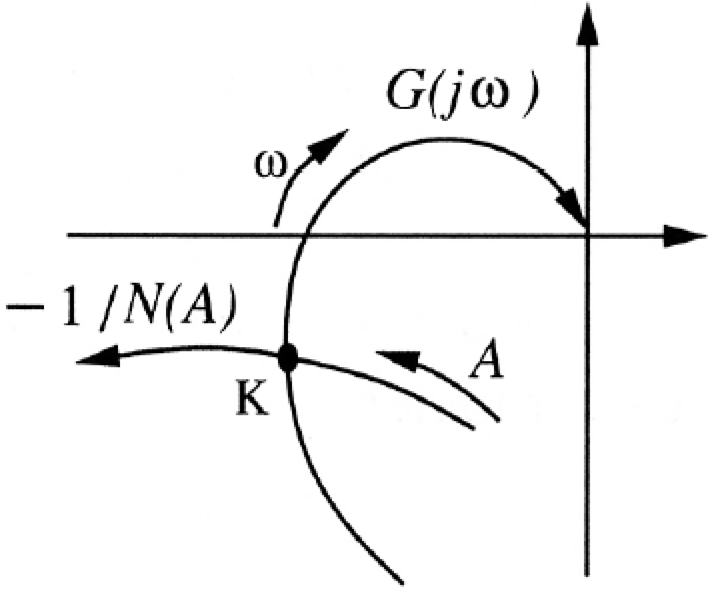


$$G(j\omega)\, N(A,\omega) + 1 = 0$$

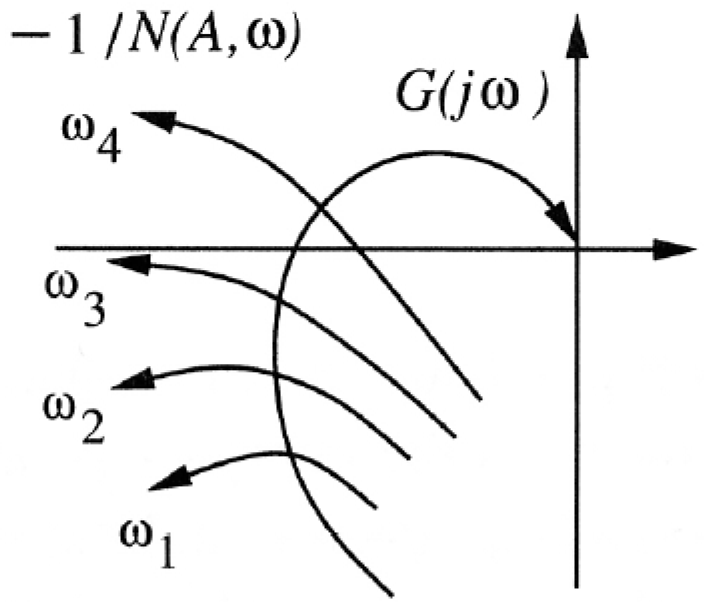
$$G(j\omega) = -\frac{1}{N(A,\omega)}$$

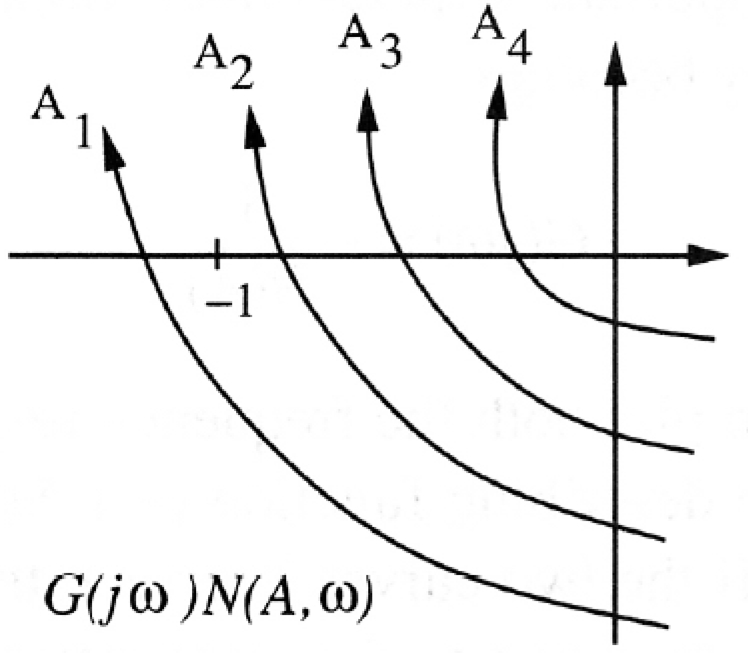
Frequency-Independent Describing Function

$$G(j\omega) = -\frac{1}{N(A)}$$



Frequency-Dependent Describing Function





Stability of Limit Cycles