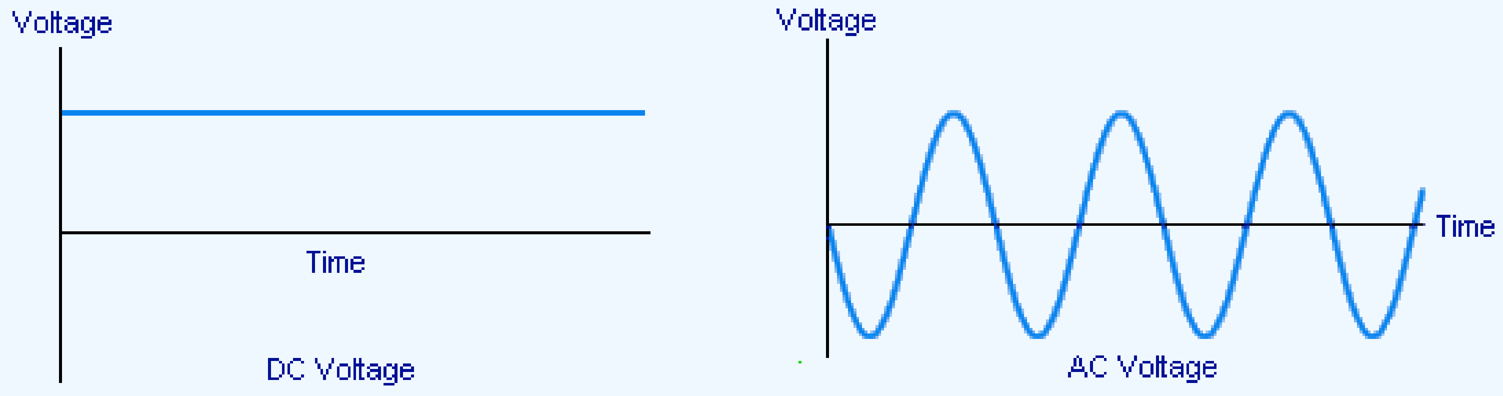
AC Circuits

Sinusoidal Steady-State Analysis

Phasor

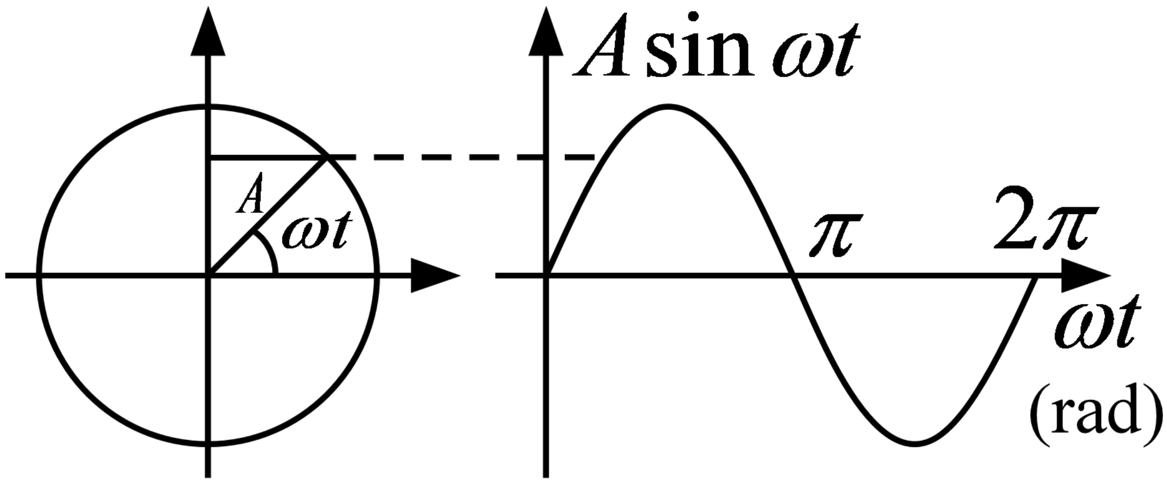
DC & AC

- DC : Direct Current
- AC : Alternating Current



Representation of Visual Difference between DC and AC voltages

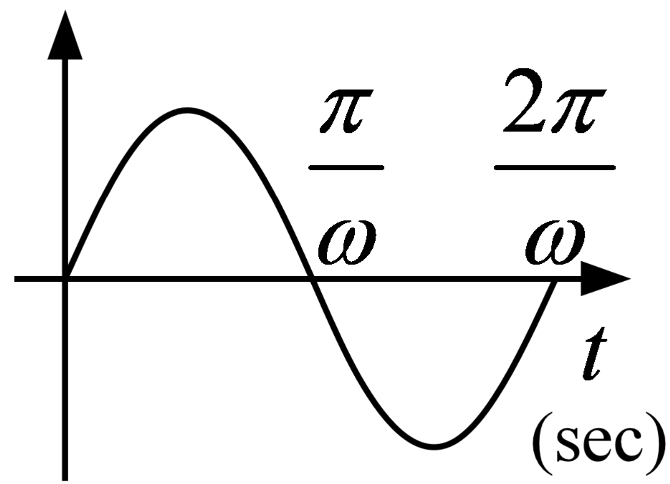
Sinusoidal function



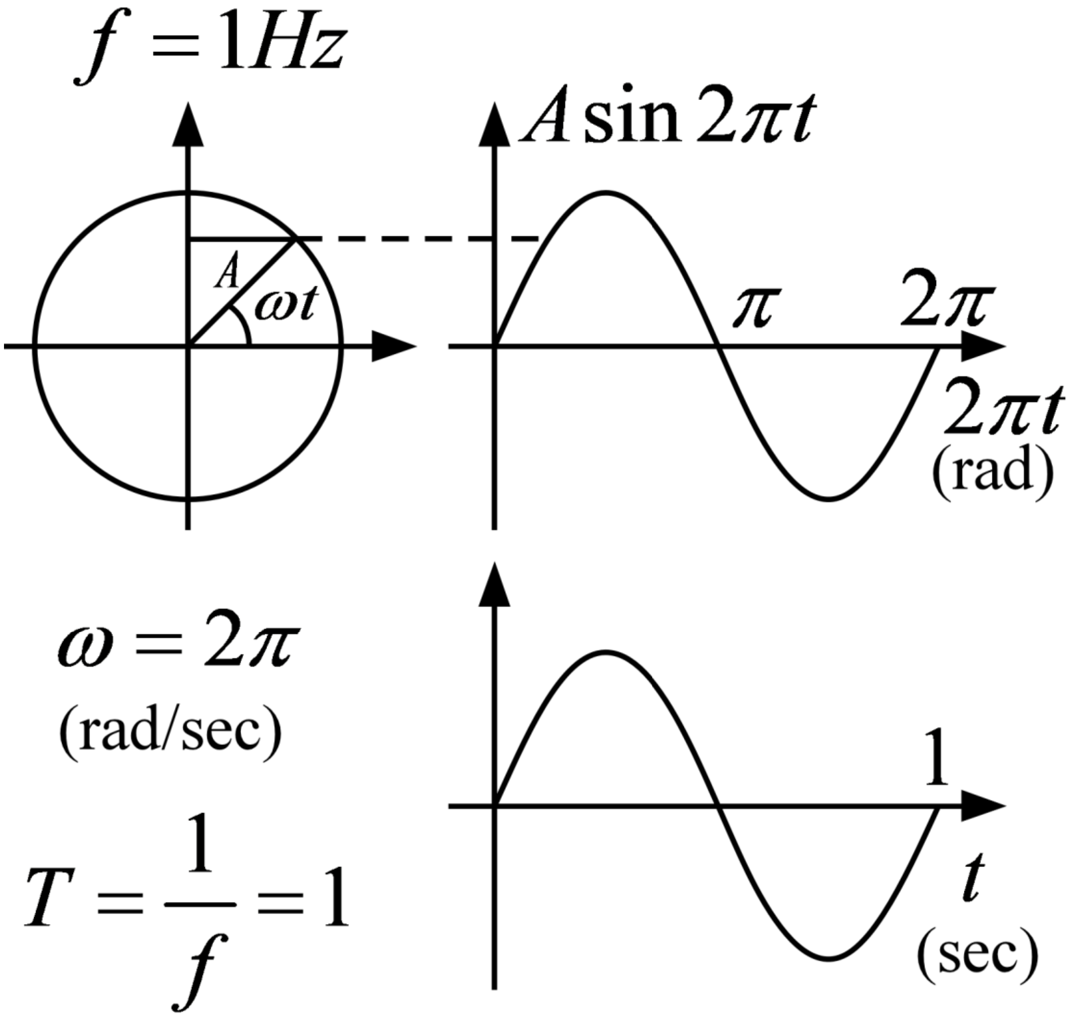
$$\omega = 2\pi f$$

(rad/sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$



Sinusoidal function



Sine and Cosine Functions

$$\sin^2 x + \cos^2 x = 1$$

$$\begin{cases} \sin(x + y) = \sin x \cos y + \cos x \sin y \\ \sin(x - y) = \sin x \cos y - \cos x \sin y \\ \cos(x + y) = \cos x \cos y - \sin x \sin y \\ \cos(x - y) = \cos x \cos y + \sin x \sin y \end{cases}$$

$$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$\begin{cases} \sin x = \cos\left(x - \frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2} - x\right) \\ \cos x = \sin\left(x + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2} - x\right) \end{cases}$$

$$\sin(\pi - x) = \sin x, \qquad \cos(\pi - x) = -\cos x$$

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x), \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

Sine and Cosine Functions

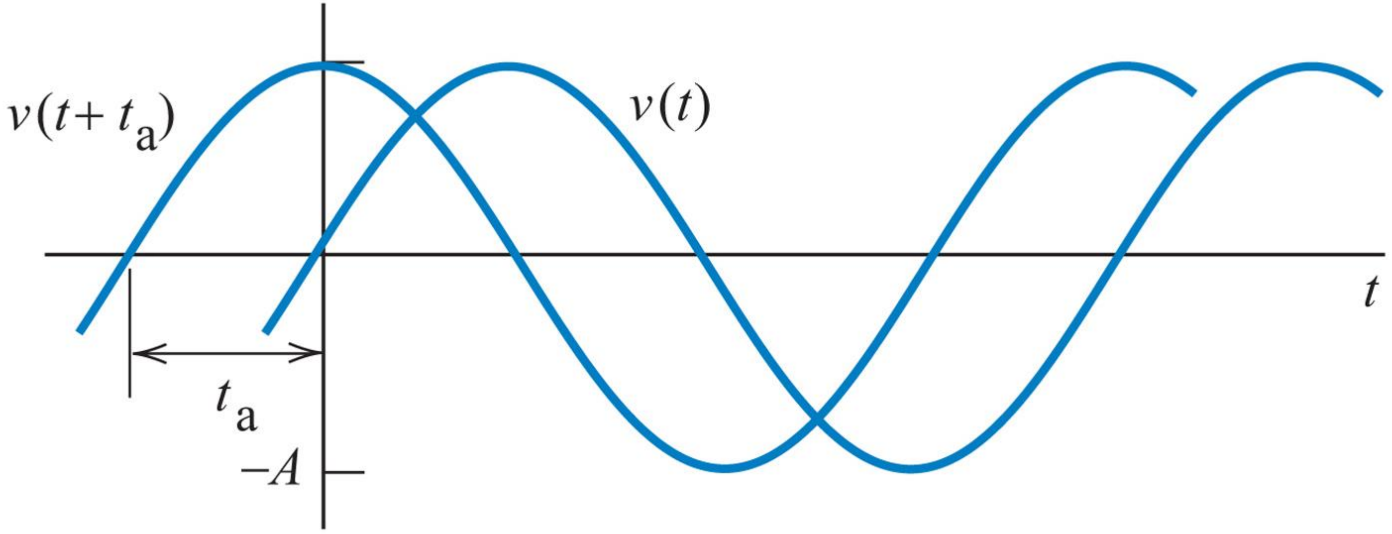
$$\begin{cases} \sin x \sin y = \frac{1}{2}[-\cos(x+y) + \cos(x-y)] \\ \cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)] \\ \sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)] \end{cases}$$

$$\begin{cases} \sin u + \sin v = 2\sin\frac{u+v}{2}\cos\frac{u-v}{2} \\ \cos u + \cos v = 2\cos\frac{u+v}{2}\cos\frac{u-v}{2} \\ \cos v - \cos u = 2\sin\frac{u+v}{2}\sin\frac{u-v}{2} \end{cases}$$

$$A\cos x + B\sin x = \sqrt{A^2+B^2}\cos(x \pm \delta), \qquad \tan\delta = \frac{\sin\delta}{\cos\delta} = \mp\frac{B}{A}$$

$$A\cos x + B\sin x = \sqrt{A^2+B^2}\sin(x \pm \delta), \qquad \tan\delta = \frac{\sin\delta}{\cos\delta} = \pm\frac{A}{B}$$

Advancing in time



$$v(t+t_a) = A\sin(\omega(t+t_a)) = A\sin(\omega t + \omega t_a) = A\sin(\omega t + \theta) \ \text{V}$$

$$\theta = \omega t_a = \frac{2\pi}{T} t_a = 2\pi \frac{t_a}{T}$$

Delaying in time

$$v(t - t_{\mathrm{d}}) = A\sin(\omega(t - t_{\mathrm{d}})) = A\sin(\omega t - \omega t_{\mathrm{d}}) = A\sin(\omega t + \theta)\ \ \mathrm{V}$$

$$\theta = -\omega t_{\mathrm{d}} = -\frac{2\pi}{T}\,t_{\mathrm{d}} = -2\pi\,\frac{t_{\mathrm{d}}}{T}$$

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EXAMPLE 10.2-1 Phase Shift and Delay

Consider the sinusoids

$$v_1(t) = 10\cos(200t + 45^\circ)\ \text{V and } v_2(t) = 8\sin(200t + 15^\circ)\ \text{V}$$

Determine the time by which $v_2(t)$ is advanced or delayed with respect to $v_1(t)$.

Solution

The two sinusoids have the same frequency but different amplitudes. The time by which $v_2(t)$ is advanced or delayed with respect to $v_1(t)$ is the time between a peak of $v_2(t)$ and the nearest peak of $v_1(t)$. The period of the sinusoids is given by

$$200 = \frac{2\pi}{T} \quad \Rightarrow \quad T = \frac{\pi}{100} = 0.0314159 = 31.4159\ \text{ms}$$

To compare the phase angles of $v_1(t)$ and $v_2(t)$ we need to express both using the same trigonometric function. Choosing cosine, represent $v_2(t)$ as

$$v_2(t) = 8\sin(200t + 15^\circ) = 8\cos(200t + 15^\circ - 90^\circ) = 8\cos(200t - 75^\circ)\ \text{V}$$

Let θ_1 and θ_2 represent the phase angles of $v_1(t)$ and $v_2(t)$. To compare $v_2(t)$ to $v_1(t)$ consider

$$\theta_2 - \theta_1 = -75^\circ - 45^\circ = -120^\circ = -\frac{\pi}{3}\ \text{rad}$$

The minus sign indicates a delay rather than an advance. Convert this angle to a time using Eq. 10.2-5

$$\theta_2 - \theta_1 = 2\pi\frac{t_d}{T} \quad \Rightarrow \quad t_d = \frac{(\theta_2 - \theta_1)T}{2\pi} = \frac{-\frac{\pi}{3}(0.0314159)}{2\pi} = -10.47\ \text{ms}$$

Again, the minus sign indicates a delay. We conclude that $v_2(t)$ is delayed with respect to $v_1(t)$ by 10.47 ms. Figure 10.2-4 shows plots of $v_1(t)$ and $v_2(t)$. (Voltage $v_1(t)$ is plotted using a dashed line and voltage $v_2(t)$ is plotted using a solid line.) Figure 10.2-4 shows that $v_2(t)$ is indeed delayed by about 10.5 ms with respect to $v_1(t)$.

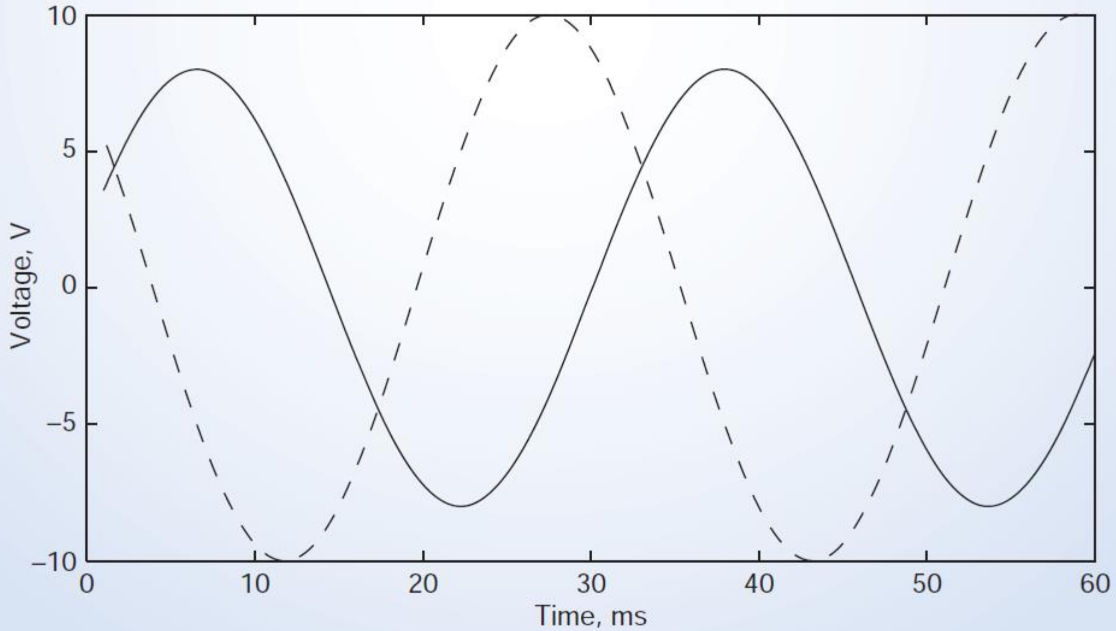
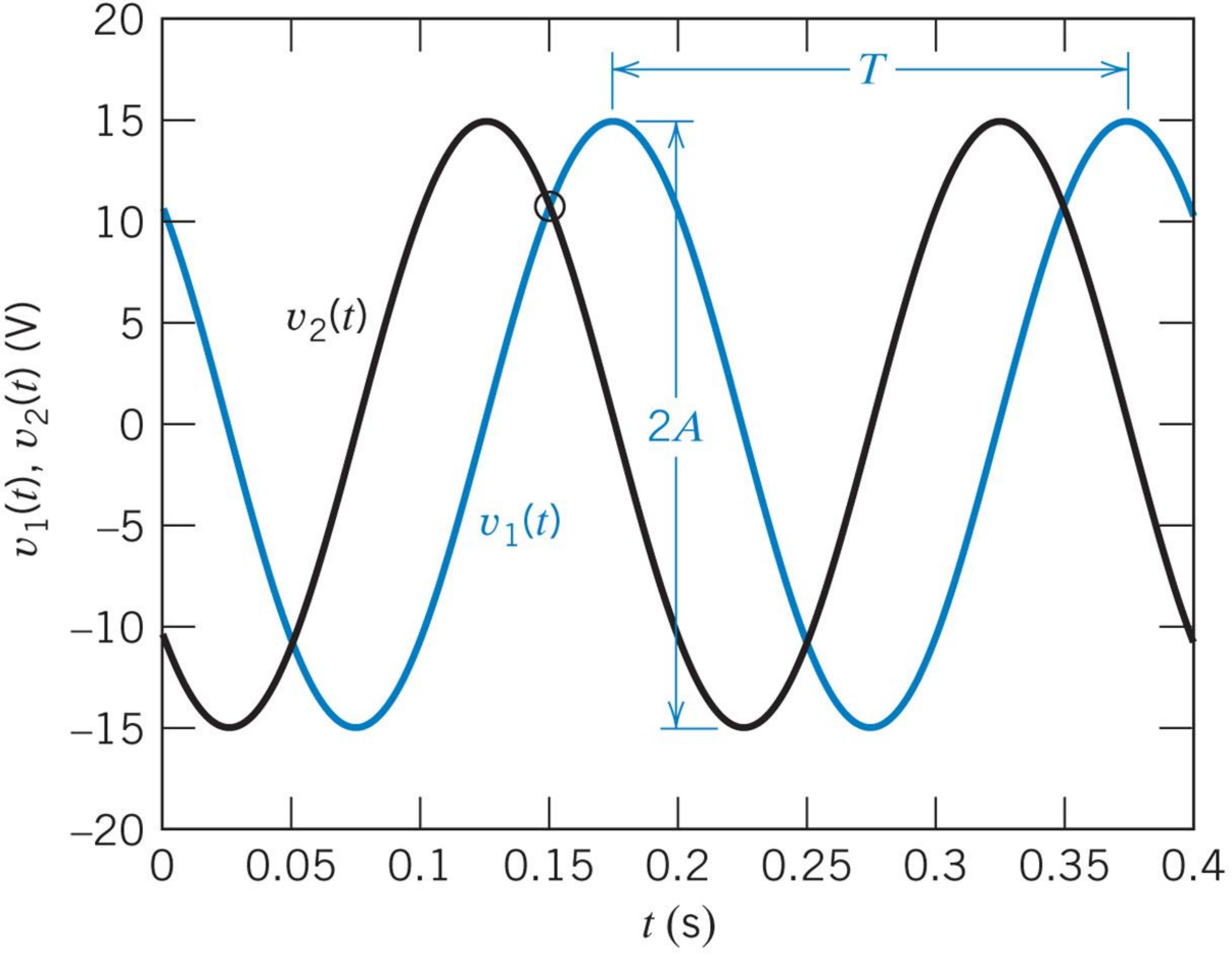
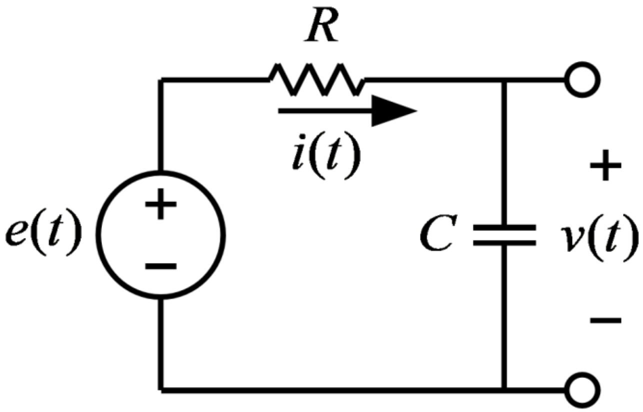


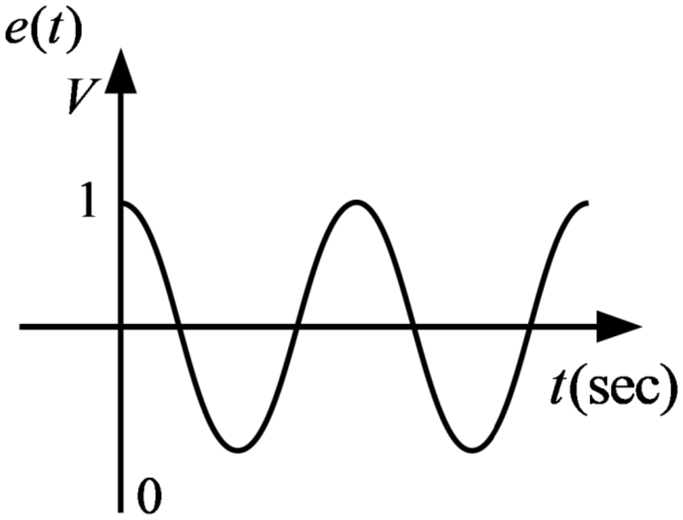
FIGURE 10.2-4 A MATLAB plot of $v_1(t)$ and $v_2(t)$ showing that $v_2(t)$ is indeed delayed with respect to $v_1(t)$ by 10.47 ms.

Oscilloscope



Differential Equation with AC Source





$$KVL:$$

$$Ri + v = e = \cos\omega t, \quad i = C\frac{dv}{dt}$$

$$RC\frac{dv}{dt} + v = \cos\omega t, \quad v(0) = 1$$

- Forced response(particular solution), steady state:

$$v = A\cos\omega t + B\sin\omega t$$

$$RC\frac{d\left(A\cos\omega t + B\sin\omega t\right)}{dt} + \left(A\cos\omega t + B\sin\omega t\right) = \cos\omega t$$

$$-\omega RCA\sin\omega t + \omega RCB\cos\omega t + A\cos\omega t + B\sin\omega t = \cos\omega t$$

$$\left(\omega RCB + A\right)\cos\omega t + \left(B - \omega RCA\right)\sin\omega t = \cos\omega$$

$$\omega RCB + A = 1, B - \omega RCA = 0$$

$$B = \omega RCA, \omega RC\left(\omega RCA\right) + A = 1$$

$$A = \frac{1}{\left(\omega RC\right)^2 + 1}, B = \frac{\omega RC}{\left(\omega RC\right)^2 + 1}$$

- Forced response(particular solution), steady state:

$$v = A\cos\omega t + B\sin\omega t$$

$$= \frac{1}{(\omega RC)^2 + 1}\cos\omega t + \frac{\omega RC}{(\omega RC)^2 + 1}\sin\omega t$$

$$= \frac{\sqrt{(\omega RC)^2 + 1}}{(\omega RC)^2 + 1}\cos(\omega t - \tan^{-1}\omega RC)$$

$$= \frac{1}{\sqrt{(\omega RC)^2 + 1}}\cos(\omega t - \tan^{-1}\omega RC)$$

- Complete response

$$v(t) = Ke^{-\frac{t}{RC}} + \frac{1}{(\omega RC)^2+1}\cos\omega t + \frac{\omega RC}{(\omega RC)^2+1}\sin\omega t$$

$$= Ke^{-\frac{t}{RC}} + \frac{1}{\sqrt{(\omega RC)^2+1}}\cos(\omega t - \tan^{-1}\omega RC)$$

$$v(0) = K + \frac{1}{(\omega RC)^2+1} = 1, K = \frac{(\omega RC)^2}{(\omega RC)^2+1}$$

$$v(t) = \frac{(\omega RC)^2}{(\omega RC)^2+1}e^{-\frac{t}{RC}} + \frac{1}{(\omega RC)^2+1}\cos\omega t + \frac{\omega RC}{(\omega RC)^2+1}\sin\omega t$$

Check Initial Condition

- Check

$$v(t)=\frac{(\omega RC)^2}{(\omega RC)^2+1}e^{-\frac{t}{RC}}+\frac{1}{(\omega RC)^2+1}\cos\omega t+\frac{\omega RC}{(\omega RC)^2+1}\sin\omega t$$

$$\frac{dv(t)}{dt}=-\frac{1}{RC}\frac{(\omega RC)^2}{(\omega RC)^2+1}e^{-\frac{t}{RC}}-\frac{\omega}{(\omega RC)^2+1}\sin\omega t+\frac{\omega^2 RC}{(RC\omega)^2+1}\cos\omega t$$

$$\left.\frac{dv(t)}{dt}\right|_{t=0}=-\frac{1}{RC}\frac{(\omega RC)^2}{(\omega RC)^2+1}+\frac{\omega^2 RC}{(\omega RC)^2+1}=0$$

Phasors and Sinusoids

A phasor is a complex number that is used to represent the amplitude and phase angle of a sinusoid. The relationship between the sinusoid and the phasor is described by

$$A\cos(\omega t + \theta) \quad \leftrightarrow \quad A \angle \theta \qquad (10.3\text{-}1)$$

- Sinusoidal source(a known fixed single frequency)
- Steady state, forced response
- Euler's formula $$e^{j\phi} = \cos\phi + j\sin\phi$$
- Time domain, Frequency domain

Euler Formula

$$e^{j\phi} = 1 + j\phi + \frac{(j\phi)^2}{2!} + \frac{(j\phi)^3}{3!} + \frac{(j\phi)^4}{4!} + \frac{(j\phi)^5}{5!} + \cdots$$

$$= 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - + \cdots + j\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - + \cdots\right)$$

$$= \cos\phi + j\sin\phi$$

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EXAMPLE 10.3-1 Phasors and Sinusoids

Determine the phasors corresponding to the sinusoids

$$i_1(t) = 120\cos(400t + 60^\circ)\ \text{mA and}\ \ i_2(t) = 100\sin(400t - 75^\circ)\ \text{mA}$$

Solution

Using Eq. 10.3-1 we have

$$\mathbf{I}_1(\omega) = 120\,\underline{/60^\circ}\ \text{mA}$$

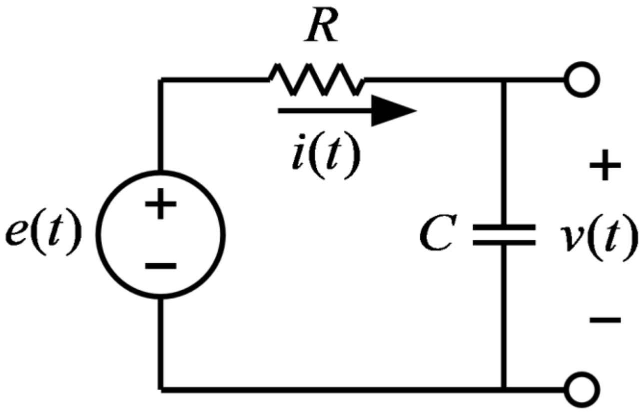
Next, express $i_2(t)$ using the cosine instead of the sine.

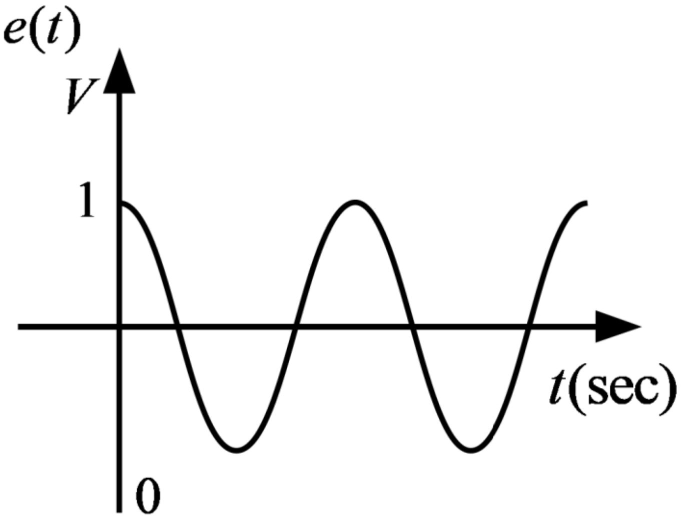
$$i_2(t) = 100\cos(400t - 75^\circ - 90^\circ) = 100\cos(400t - 165^\circ)\ \text{mA}$$

(See the trigonometric identities in Appendix C.) Using Eq. 10.3-1, we have

$$\mathbf{I}_2(\omega) = 100\,\underline{/-165^\circ}\ \text{mA}$$

Motivating Example: RC circuit





$$KVL:$$

$$Ri + v = e = \cos\omega t, \quad i = C\frac{dv}{dt}$$

$$RC\frac{dv}{dt} + v = \cos\omega t, \quad v(0) = 1$$

Motivating Example: RC circuit

- Euler's formula

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$\cos\omega t = \mathrm{Re}\left\{e^{j\omega t}\right\}$$

$$RC\frac{dv}{dt} + v = \cos\omega t \quad \Rightarrow \quad RC\frac{dv}{dt} + v = e^{j\omega t}$$

- Assume the solution: $v = Ve^{j(\omega t+\phi)} = Ve^{j\phi}e^{j\omega t}$

Motivating Example: RC circuit

- Assume the solution: $v = Ve^{j(\omega t+\phi)} = Ve^{j\phi}e^{j\omega t}$

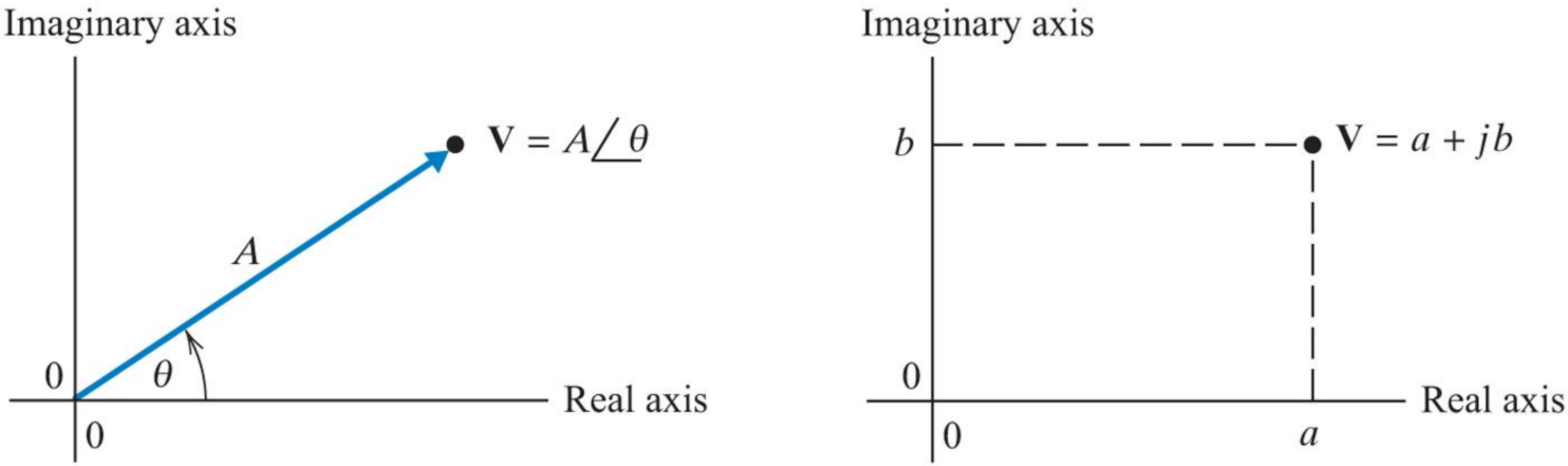
$$RC\frac{dVe^{j(\omega t+\phi)}}{dt} + Ve^{j(\omega t+\phi)} = e^{j\omega t}$$

$$j\omega RCVe^{j(\omega t+\phi)} + Ve^{j(\omega t+\phi)} = e^{j\omega t} \Rightarrow j\omega RCVe^{j\phi}e^{j\omega t} + Ve^{j\phi}e^{j\omega t} = e^{j\omega t}$$

$$j\omega RCVe^{j\phi} + Ve^{j\phi} = 1 \Rightarrow \left(j\omega RC + 1\right)Ve^{j\phi} = 1$$

$$Ve^{j\phi} = \frac{1}{\left(j\omega RC + 1\right)} = \frac{1}{\sqrt{\left(\omega RC\right)^2 + 1}}e^{-\tan^{-1}\omega RC}$$

$$\mathrm{Re}\left\{Ve^{j(\omega t+\phi)}\right\} = \frac{1}{\sqrt{\left(RC\omega\right)^2 + 1}}\cos(\omega t - \tan^{-1}RC\omega)$$



(a) (b)

$$A = |\mathbf{V}| \text{ and } \theta = \angle \mathbf{V}$$

$$a = \text{Re}\{\mathbf{V}\} \text{ and } b = \text{Im}\{\mathbf{V}\}$$

$$a + jb = \mathbf{V} = A \angle \theta$$

$$a = A\cos(\theta),\ b = A\sin(\theta),\ A = \sqrt{a^2 + b^2}$$

$$\theta = \begin{cases} \tan^{-1}\left(\dfrac{b}{a}\right) & a > 0 \\ 180^\circ - \tan^{-1}\left(\dfrac{b}{-a}\right) & a < 0 \end{cases}$$

Arithmetic using Phasors

$$\mathbf{V}_1 + \mathbf{V}_2 = (a + j\,b) + (c + j\,d) = (a + c) + j\,(b + d)$$

$$\mathbf{V}_1 - \mathbf{V}_2 = (a + j\,b) - (c + j\,d) = (a - c) + j\,(b - d)$$

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = \left(E\angle\theta\right)\left(F\angle\phi\right) = EF\angle(\theta + \phi) \text{ and } \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{A\angle\theta}{B\angle\phi} = \frac{A}{B}\angle(\theta - \phi)$$

$$\mathbf{V}_1^{\,*} = (a + jb)^* = a - jb$$
$$= \left(E\angle\theta\right)^* = E\angle{-\theta}$$

$$A\,e^{j(\omega t+\theta)} = A\cos(\omega t+\theta) + j\,A\sin(\omega t+\theta)$$

$$A\cos(\omega t+\theta) = \mathrm{Re}\left\{A\,e^{j(\omega t+\theta)}\right\} = \mathrm{Re}\left\{A\,e^{j\theta}\,e^{j\omega t}\right\}$$

$$v(t) = A\cos(\omega t+\theta)\ \ \mathrm{V}\ \ \text{and}\ \ \mathbf{V}(\omega) = A\angle\theta = A\,e^{j\theta}\ \ \mathrm{V}$$

$$v(t) = \mathrm{Re}\left\{\mathbf{V}(\omega)\,e^{j\omega t}\right\}$$

KVL in Phasor

$$0 = \sum_i v_i(t) \qquad (10.3\text{-}18)$$

Using Eq. 10.3-17, we can write Eq. 10.3-18 as

$$0 = \sum_i \text{Re}\{\mathbf{V}_i(\omega)\, e^{j\omega t}\} = \text{Re}\left\{e^{j\omega t} \sum_i \mathbf{V}_i(\omega)\right\} \qquad (10.3\text{-}19)$$

Eq. 10.3-19 is required to be true for all values of time t. Let $t = 0$. Then $e^{j\omega t} = e^{0} = 1$ and Eq. 10.3-19 becomes

$$0 = \text{Re}\left\{\sum_i \mathbf{V}_i(\omega)\right\} \qquad (10.3\text{-}20)$$

Next, let $t = \pi/(2\,\omega)$. Then $e^{j\omega t} = e^{-j\pi/2} = -j$ and Eq. 10.3-19 becomes

$$0 = \text{Re}\left\{-j \sum_i \mathbf{V}_i(\omega)\right\} = \text{Im}\left\{\sum_i \mathbf{V}_i(\omega)\right\} \qquad (10.3\text{-}21)$$

Together, Eqs. 10.3-19 and 10.3-21 indicate that the phasors 0 and $\sum_i \mathbf{V}_i(\omega)$ are equal. That is,

$$0 = \sum_i \mathbf{V}_i(\omega)$$

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EXAMPLE 10.3-4 Kirchhoff's Laws for AC Circuits

The input to the circuit shown in Figure 10.3-3 is the voltage source voltage,

$$v_s(t) = 25\cos(100t + 15°)\ \text{V}$$

The output is the voltage across the capacitor,

$$v_C(t) = 20\cos(100t - 22°)\ \text{V}$$

Determine the resistor voltage $v_R(t)$.

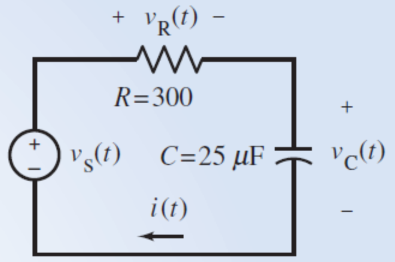


FIGURE 10.3-3 The circuit in Example 10.3-4

Solution

Apply KVL to get

$$v_R(t) = v_s(t) - v_C(t) = 25\cos(100t + 15°) - 20\cos(100t - 22°)$$

Writing the KVL equation using phasors, we have

$$\begin{aligned}\mathbf{V}_R(\omega) = \mathbf{V}_s(\omega) - \mathbf{V}_C(\omega) &= 25\angle 15° - 20\angle -22° \\ &= (24.15 + j6.47) - (18.54 - j7.49) \\ &= 5.61 + j13.96 \\ &= 15\angle 68.1°\ \text{V}\end{aligned}$$

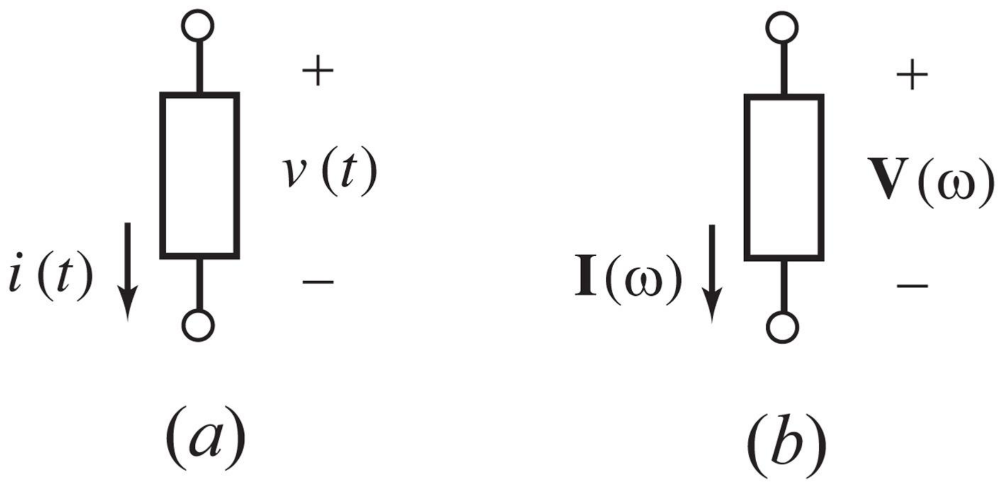
Converting the phasor $\mathbf{V}_R(\omega)$ to the corresponding sinusoid, we have

$$\mathbf{V}_R(\omega) = 15\angle 68.1°\ \text{V} \quad \Leftrightarrow \quad v_R(t) = 15\cos(100t + 68.1°)\ \text{V}$$

Impedances

The impedance of an element of an ac circuit is defined to be the ratio of the voltage phasor to the current phasor. The impedance is denoted as $\mathbf{Z}(\omega)$ so

$$\mathbf{Z}(\omega) = \frac{\mathbf{V}(\omega)}{\mathbf{I}(\omega)} = \frac{V_m \angle\theta}{I_m \angle\phi} = \frac{V_m}{I_m} \angle(\theta - \phi)\ \Omega \tag{10.4-2}$$



Ohm's Law for AC circuits

- Impedance and admittance

$$V(\omega) = Z(\omega)I(\omega)$$

$$Y(\omega) = \frac{1}{Z(\omega)} = \frac{I(\omega)}{V(\omega)}$$

Capacitor

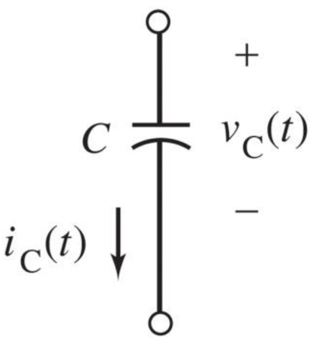
$$v_C(t) = A\cos(\omega t + \theta) \;\; \text{V}$$

$$i_C(t) = C\frac{d}{dt}v_C(t) = -C\omega A\sin(\omega t + \theta) = C\omega A\cos(\omega t + \theta + 90°) \;\; \text{A}$$

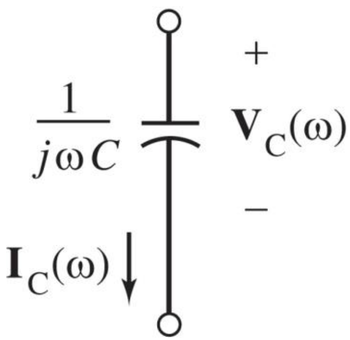
The phasors corresponding to the capacitor voltage and current are

$$\mathbf{V}_\text{C}(\omega) = A\,\angle\underline{\theta} \;\; \text{V and } \mathbf{I}_\text{C}(\omega) = C\omega A\,\angle\underline{(\theta + 90°)} = \left(C\omega\,\angle\underline{90°}\right)\left(A\,\angle\underline{\theta}\right) = j\omega CA\,\angle\underline{\theta} \;\; \text{A}$$

The impedance of the capacitor is given by the ratio of the voltage phasor to the current phasor:



(a)



(b)

$$\mathbf{Z}_\text{C}(\omega) = \frac{\mathbf{V}_\text{C}(\omega)}{\mathbf{I}_\text{C}(\omega)} = \frac{A\,\angle\underline{\theta}}{j\omega CA\,\angle\underline{\theta}} = \frac{1}{j\omega C} \;\; \Omega$$

$$\mathbf{V}_\text{C}(\omega) = \frac{1}{j\omega C}\mathbf{I}_\text{C}(\omega)$$

Inductor

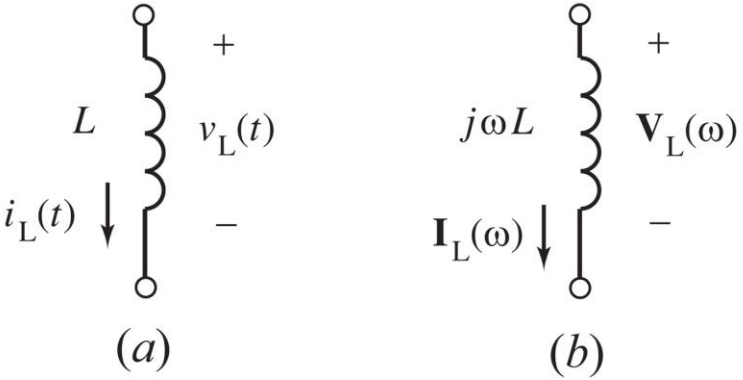
$$i_L(t) = A\cos(\omega t + \theta) \ \ \text{A}$$

$$v_L(t) = L\frac{d}{dt}i_L(t) = -L\omega A\sin(\omega t + \theta) = L\omega A\cos(\omega t + \theta + 90°) \ \ \text{V}$$

The phasors corresponding to the inductor current and voltage are

$$\mathbf{I}_\text{L}(\omega) = A\angle\theta \ \text{A and } \mathbf{V}_\text{L}(\omega) = L\omega A\angle(\theta + 90°) = j\omega LA\angle\theta \ \text{V}$$

The impedance of the inductor is given by the ratio of the voltage phasor to the current phasor:



$$\mathbf{Z}_\text{L}(\omega) = \frac{\mathbf{V}_\text{L}(\omega)}{\mathbf{I}_\text{L}(\omega)} = \frac{j\omega LA\angle\theta}{A\angle\theta} = j\omega L \ \Omega$$

$$\mathbf{V}_\text{L}(\omega) = j\omega L\mathbf{I}_\text{L}(\omega)$$

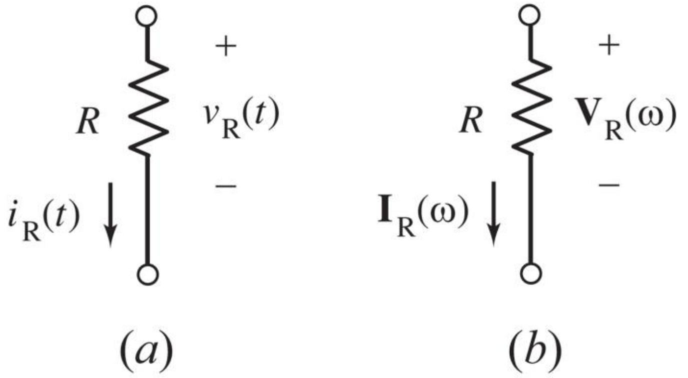
Resistor

$$v_R(t) = A\cos(\omega t + \theta)$$

$$i_R(t) = \frac{v_R(t)}{R} = \frac{A}{R}\cos(\omega t + \theta)$$

The impedance of the resistor is the ratio of the voltage phasor to the current phasor:

$$\mathbf{Z}_R(\omega) = \frac{\mathbf{V}_R(\omega)}{\mathbf{I}_R(\omega)} = \frac{A\angle\theta}{\frac{A}{R}\angle\theta} = R\ \Omega$$



$$\mathbf{V}_R(\omega) = R\,\mathbf{I}_R(\omega)$$

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EXAMPLE 10.4-1 Impedances

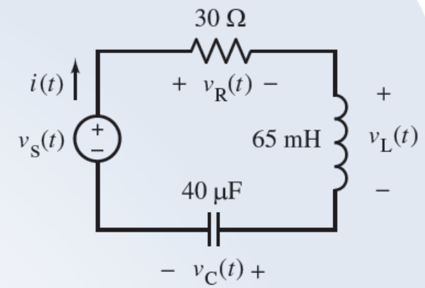


FIGURE 10.4-5 The AC circuit in Eample 10.4-1.

The input to the ac circuit shown in Figure 10.4-5 is the source voltage

$$v_S(t) = 12\cos(1000t + 15°)\ \text{V}$$

Determine (a) the impedances of the capacitor, inductor, and resistance and (b) the current $i(t)$.

Solution

(a) The input frequency is $\omega = 1000$ rad/s. Using Eq. 10.4-4 shows that the impedance of the capacitor is

$$\mathbf{Z}_C(\omega) = \frac{1}{j\omega C} = \frac{1}{j1000\,(40 \times 10^{-6})} = \frac{25}{j} = -j25\ \Omega$$

Using Eq. 10.4-6 shows that the impedance of the inductor is

$$\mathbf{Z}_L(\omega) = j\omega L = j1000\,(0.065) = j65\ \Omega$$

Using Eq. 10.4-8, the impedance of the resistor is

$$\mathbf{Z}_R(\omega) = R = 30\ \Omega$$

(b) Apply KVL to write

$$12\cos(1000t + 15°) = v_R(t) + v_L(t) + v_C(t)$$

Using phasors, we get

$$12\angle 15° = \mathbf{V}_R(\omega) + \mathbf{V}_L(\omega) + \mathbf{V}_C(\omega) \qquad (10.4\text{-}10)$$

Using Eqs. 10.4-5, 10.4-7, and 10.4-9, we get

$$12\angle 15° = 30\,\mathbf{I}(\omega) + j65\,\mathbf{I}(\omega) - j25\,\mathbf{I}(\omega) = (30 + j40)\,\mathbf{I}(\omega) \qquad (10.4\text{-}11)$$

Solving for $\mathbf{I}(\omega)$ gives

$$\mathbf{I}(\omega) = \frac{12\angle 15°}{30 + j40} = \frac{12\angle 15°}{50\angle 53.13°} = 0.24\angle 38.13°\ \text{A}$$

The corresponding sinusoid is

$$i(t) = 0.24\cos(1000t - 38.13°)\ \text{A}$$

Try it yourself in WileyPLUS

EXAMPLE 10.4-2 AC Circuits in the Frequency Domain

The input to the ac circuit shown in Figure 10.4-7 is the source voltage

$$v_S(t) = 48\cos(500t + 75^\circ)\ \text{V}$$

Determine the voltage $v(t)$.

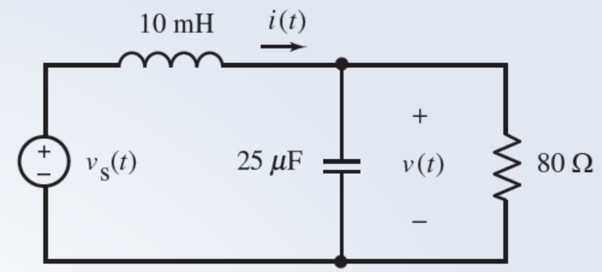


FIGURE 10.4-7 The ac circuit in Example 10.4-2.

Solution

The input frequency is $\omega = 500$ rad/s. The impedances of the capacitor and inductor are

$$\mathbf{Z}_C(\omega) = \frac{1}{j\omega C} = \frac{1}{j500\left(25\times 10^{-6}\right)} = \frac{80}{j} = -j80\ \Omega,$$

and

$$\mathbf{Z}_L(\omega) = j\omega L = j500(0.1) = j50\ \Omega$$

Figure 10.4-8 shows the circuit represented in the frequency domain using phasors and impedances. Notice that

(a) The voltage source voltage is described by the phasor corresponding to $v_s(t)$.

(b) The currents and voltages of the resistor, inductor, and capacitor are described by the phasors $\mathbf{I}_R(\omega)$, $\mathbf{V}(\omega)$, $\mathbf{I}(\omega)$, $\mathbf{V}_L(\omega)$, and $\mathbf{I}_C(\omega)$.

(c) The resistor, inductor, and capacitor are described by their impedances.

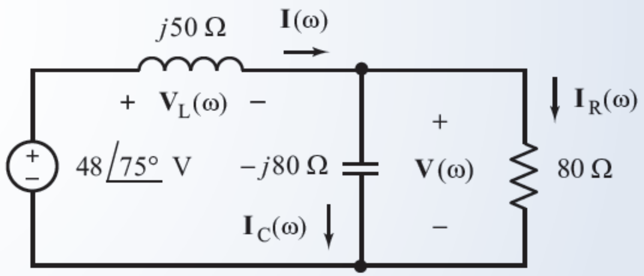
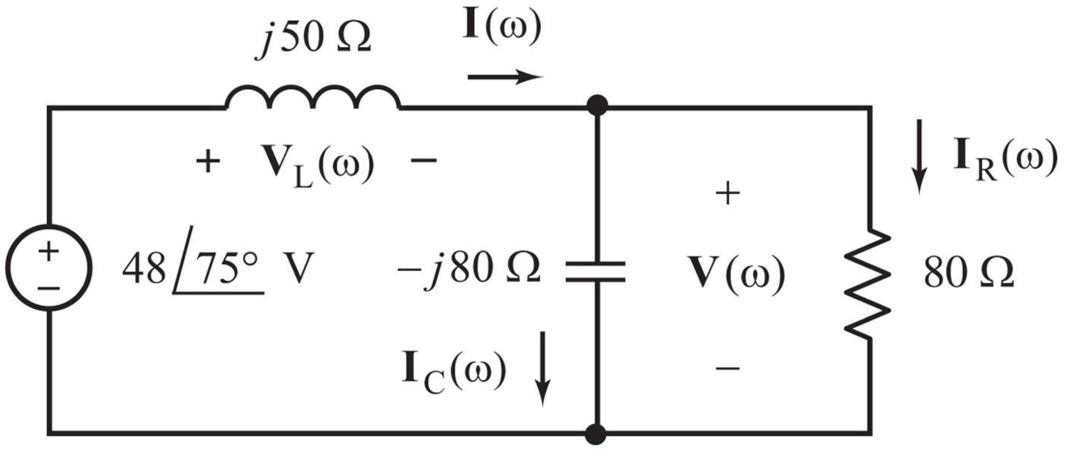


FIGURE 10.4-8 The circuit from Figure 10.4-7, represented in the frequency domain.

Node analysis



$$\text{KCL}: I(\omega) = I_C(\omega) + I_R(\omega)$$

$$\frac{48\angle 75° - V(\omega)}{j50} = \frac{V(\omega)}{-j80} + \frac{V(\omega)}{80}$$

$$48\angle 75° = j50\left(\frac{V(\omega)}{-j80} + \frac{V(\omega)}{80}\right) + V(\omega) = \left(\frac{j50}{-j80} + \frac{j50}{80} + 1\right)V(\omega)$$

Node analysis(MATLAB)

$$48\angle 75^\circ = \left(\frac{j50}{-j80} + \frac{j50}{80} + 1 \right) V(\omega)$$

```
>> V=48*exp(j*75*pi/180)/(j*50/(-j*80)+(j*50)/80+1)
V =
  63.3158 +18.1122i

>> abs(V)
ans =
   65.8555

>> angle(V)*180/pi
ans =
   15.9638
```

Apply Ohm's law to each of the impedances to get

$$\mathbf{V_L}(\omega) = j50\,\mathbf{I}(\omega), \quad \mathbf{I}_C(\omega) = \frac{\mathbf{V}(\omega)}{-j80} \text{ and } \mathbf{I}_R(\omega) = \frac{\mathbf{V}(\omega)}{80}$$

Apply KCL to the top left node to get

$$\mathbf{I}(\omega) = \mathbf{I}_C(\omega) + \mathbf{I}_R(\omega) = \frac{\mathbf{V}(\omega)}{-j80} + \frac{\mathbf{V}(\omega)}{80} \tag{10.4-12}$$

Apply KVL to the right mesh to get

$$48\angle 75^\circ = \mathbf{V_L}(\omega) + \mathbf{V}(\omega) = j50\,\mathbf{I}(\omega) + \mathbf{V}(\omega) \tag{10.4-13}$$

Combining Eqs. 10.4-12 and 10.4-13 gives

$$\begin{aligned}48\angle 75^\circ = j50\left[\frac{\mathbf{V}(\omega)}{-j80} + \frac{\mathbf{V}(\omega)}{80}\right] + \mathbf{V}(\omega) &= \left[\frac{j50}{-j80} + \frac{j50}{80} + 1\right]\mathbf{V}(\omega) \\ &= [-0.625 + j0.625 + 1]\mathbf{V}(\omega) = (0.375 + j0.625)\mathbf{V}(\omega)\end{aligned}$$

Solving for $\mathbf{V}(\omega)$ gives

$$\mathbf{V}(\omega) = \frac{48\angle 75^\circ}{0.375 + j0.625} = \frac{48\angle 75^\circ}{0.7289\angle 59^\circ} = 65.9\angle 16^\circ \text{ V}$$

The corresponding sinusoid is

$$v(t) = 65.9\cos(500t + 16^\circ) \text{ V}$$

Series Impedance

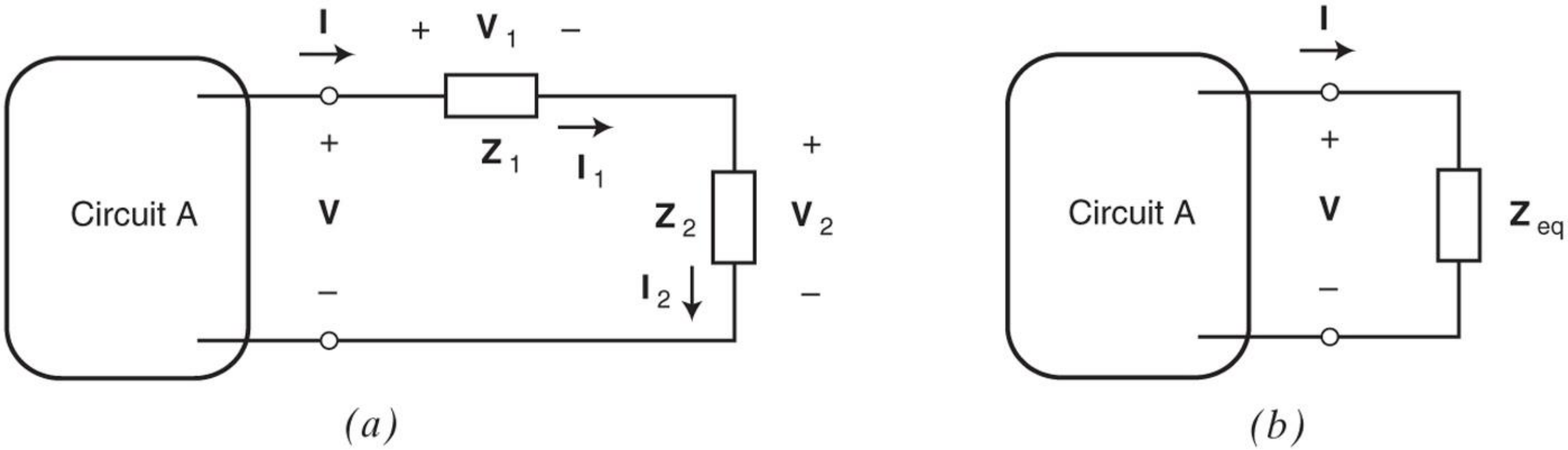
$$\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I}$$

$$\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}_1 = \mathbf{Z}_1\mathbf{I} \text{ and } \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}_2 = \mathbf{Z}_2\mathbf{I}$$

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = (\mathbf{Z}_1 + \mathbf{Z}_2)\mathbf{I}$$

$$\frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2$$

$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2$$



$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n$$

$$\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I} = \mathbf{Z}_1\frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V} \text{ and } \mathbf{V}_2 = \mathbf{Z}_2\mathbf{I} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$$

Parallel Impedance

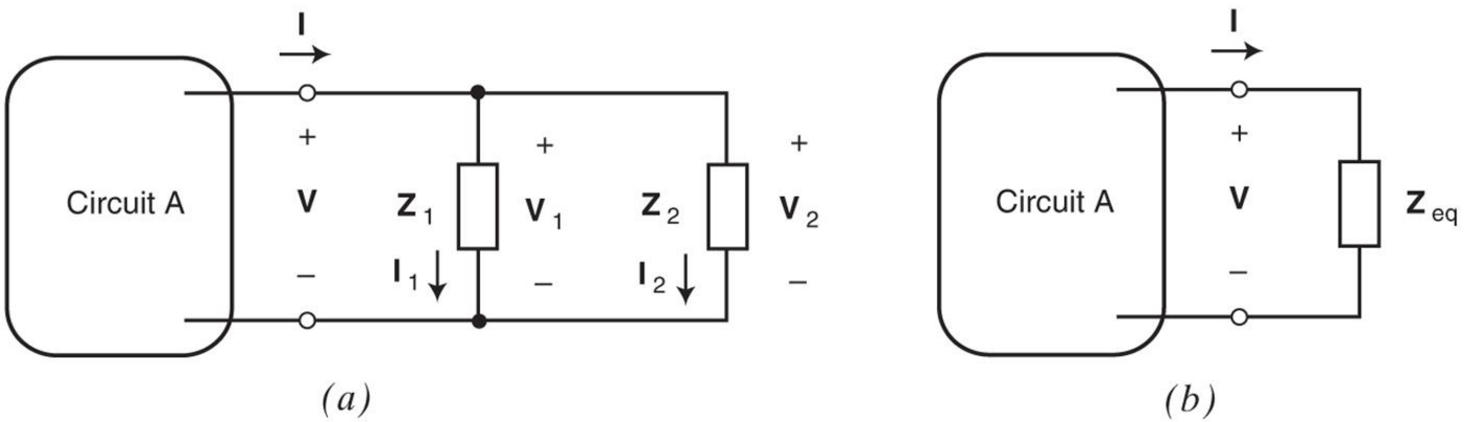
$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}}$$

$$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$$

$$\mathbf{I}_1 = \frac{\mathbf{V}_1}{\mathbf{Z}_1} = \frac{\mathbf{V}}{\mathbf{Z}_1} \text{ and } \mathbf{I}_2 = \frac{\mathbf{V}_2}{\mathbf{Z}_2} = \frac{\mathbf{V}}{\mathbf{Z}_2}$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}\right)\mathbf{V} \qquad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}}$$

$$Z_{eq} = Z_1 \parallel Z_2 = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

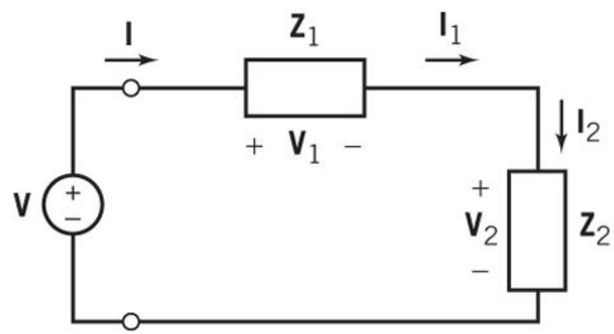
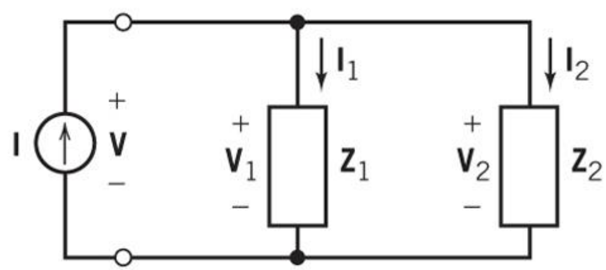


$$\mathbf{Z}_{eq} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n}} \qquad \mathbf{Y}_{eq} = \frac{1}{\mathbf{Z}_{eq}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_n} = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n$$

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1} = \frac{1}{\mathbf{Z}_1}\,\frac{\mathbf{I}}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2}} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \text{ and } \mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2} = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$$

Voltage and Current Division

- In the frequency domain

	CIRCUIT	EQUATIONS
Voltage division		$\mathbf{I}_1 = \mathbf{I}_2 = \mathbf{I}$ $\mathbf{V}_1 = \dfrac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$ $\mathbf{V}_2 = \dfrac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V}$
Current division		$\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$ $\mathbf{I}_1 = \dfrac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$ $\mathbf{I}_2 = \dfrac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I}$

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EXAMPLE 10.5-1 Analysis of AC Circuits Using Impedances

Determine the steady-state current $i(t)$ in the *RLC* circuit shown in Figure 10.5-3*a*, using phasors and impedances.

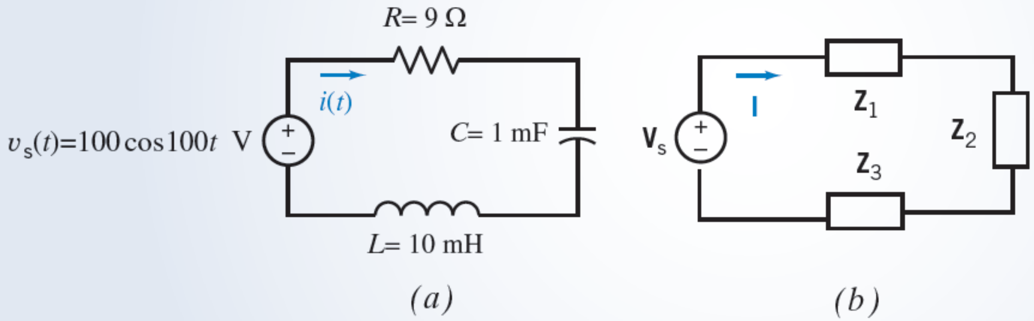

FIGURE 10.5-3 The circuit from Example 10.5-1 represented (*a*) in the time domain and (*b*) in the frequency domain.

Solution

First, we represent the circuit in using phasors and impedances as shown in Figure 10.5-3*b*. Noticing that the frequency of the sinusoidal input in Figure 10.5-3*a* is $\omega = 100$ rad/s, the impedances in Figure 10.5-3*b* are determined to be

$$\mathbf{Z}_1 = R = 9\,\Omega,\ \mathbf{Z}_2 = \frac{1}{j\omega C} = \frac{1}{j(100)(0.001)} = \frac{10}{j} = -j10\,\Omega$$

and

$$\mathbf{Z}_3 = j\omega L = j(100)(0.001) = j1\,\Omega$$

The input phasor in Figure 10.5-3*b* is

$$\mathbf{V}_s = 100\,\angle 0^\circ\ \text{V}$$

Solution

First, we represent the circuit in using phasors and impedances as shown in Figure 10.5-3*b*. Noticing that the frequency of the sinusoidal input in Figure 10.5-3*a* is $\omega = 100$ rad/s, the impedances in Figure 10.5-3*b* are determined to be

$$\mathbf{Z}_1 = R = 9\,\Omega,\ \mathbf{Z}_2 = \frac{1}{j\omega C} = \frac{1}{j(100)(0.001)} = \frac{10}{j} = -j10\,\Omega$$

and

$$\mathbf{Z}_3 = j\omega L = j(100)(0.001) = j1\,\Omega$$

The input phasor in Figure 10.5-3*b* is

$$\mathbf{V}_s = 100\,\angle 0^\circ\ \mathrm{V}$$

Next, we use KVL in Figure 10.5-3*b* to obtain

$$\mathbf{Z}_1\mathbf{I} + \mathbf{Z}_2\mathbf{I} + \mathbf{Z}_3\mathbf{I} = \mathbf{V}_s$$

Substituting for the impedances and the input phasor gives

$$(9 - j10 + j1)\mathbf{I} = 100\,\angle 0^\circ$$

or

$$\mathbf{I} = \frac{100\,\angle 0^\circ}{9 - j9} = \frac{10\,\angle 0^\circ}{9\sqrt{2}\,\angle -45^\circ} = 7.86\,\angle 45^\circ\ \mathrm{A}$$

Therefore, the steady-state current in the time domain is

$$i(t) = 7.86\cos(100t + 45^\circ)\ \mathrm{A}$$

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EXAMPLE 10.5-2 Voltage Division Using Impedances

INTERACTIVE EXAMPLE

Consider the circuit shown in Figure 10.5-4*a*. The input to the circuit is the voltage of the voltage source,

$$v_s(t) = 7.28\cos(4t + 77^\circ)\ \text{V}$$

The output is the voltage across the inductor $v_o(t)$. Determine the steady-state output voltage $v_o(t)$.

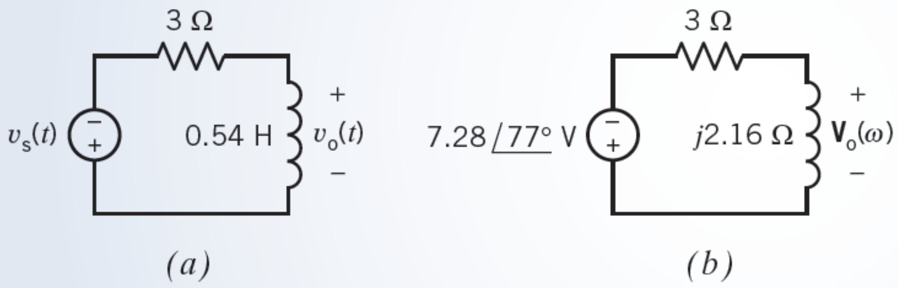

FIGURE 10.5-4 The circuit considered in Example 10.5-2 represented (*a*) in the time domain and (*b*) in the frequency domain.

Solution

The input voltage is sinusoid. The output voltage is also sinusoid and has the same frequency as the input voltage. The circuit has reached steady state. Consequently, the circuit in Figure 10.5-4*a* can be represented in the frequency domain, using phasors and impedances. Figure 10.5-4*b* shows the frequency-domain representation of the circuit from Figure 10.5-4*a*. The impedance of the inductor is $j\omega L = j(4)(0.54) = j2.16\ \Omega$, as shown in Figure 10.5-4*b*.

Apply the voltage divider principle to the circuit in Figure 10.5-4*b* to represent the output voltage in the frequency domain as

$$\mathbf{V}_o(\omega) = \frac{j2.16}{3 + j2.16}\left(-7.28\,\angle 77^\circ\right) = \frac{2.16\,\angle 90^\circ}{3.70\,\angle 36^\circ}\left(-7.28\,\angle 77^\circ\right)$$

$$= \frac{(2.16)(-7.28)}{3.70}\,\angle (90^\circ + 77^\circ) - 36^\circ$$

$$= -4.25\,\angle 131^\circ = 4.25\,\angle 311^\circ \text{ V}$$

In the time domain, the output voltage is represented as

$$v_o(t) = 4.25\cos(4t + 311^\circ)\text{ V}$$

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EXAMPLE 10.5-3 AC Circuit Analysis

INTERACTIVE EXAMPLE

Consider the circuit shown in Figure 10.5-5*a*. The input to the circuit is the voltage of the voltage source,

$$v_s(t) = 7.68\cos(2t + 47°)\ \text{V}$$

The output is the voltage across the resistor,

$$v_o(t) = 1.59\cos(2t + 125°)\ \text{V}$$

Determine capacitance C of the capacitor.

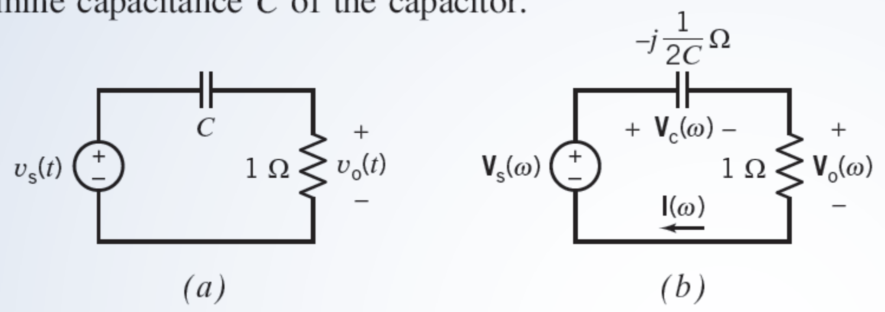

FIGURE 10.5-5 The circuit considered in Example 10.5-3 represented (*a*) in the time domain and (*b*) in the frequency domain.

Solution

The input voltage is sinusoid. The output voltage is also sinusoid and has the same frequency as the input voltage. Apparently, the circuit has reached steady state. Consequently, the circuit in Figure 10.5-5*a* can be represented in the frequency domain, using phasors and impedances. Figure 10.5-5*b* shows the frequency-domain representation of the circuit from Figure 10.5-5*a*. The impedance of the capacitor is

$$\frac{1}{j\omega C} = \frac{j}{j^2\omega C} = -\frac{j}{\omega C} = -\frac{j}{2\,C}$$

The phasors corresponding to the input and output sinusoids are

$$\mathbf{V}_{\mathrm{s}}(\omega) = 7.68\,\angle 47^\circ\ \mathrm{V}$$

and

$$\mathbf{V}_{\mathrm{o}}(\omega) = 1.59\,\angle 125^\circ\ \mathrm{V}$$

The current $\mathbf{I}(\omega)$ in Figure 10.5-5*b* is given by

$$\mathbf{I}(\omega) = \frac{\mathbf{V}_{\mathrm{o}}(\omega)}{1} = \frac{1.59\,\angle 125^\circ}{1\,\angle 0^\circ} = 1.59\,\angle 125^\circ\ \mathrm{A}$$

The capacitor voltage $\mathbf{V}_c(\omega)$ in Figure 10.5-5*b* is given by

$$\begin{aligned}\mathbf{V}_c(\omega) = \mathbf{V}_s(\omega) - \mathbf{V}_o(\omega) &= 7.68\angle 47^\circ - 1.59\angle 125^\circ \\ &= (5.23 + j5.62) - (-0.91 + 1.30) \\ &= (5.23 + 0.91) + j(5.62 - 1.30) \\ &= 6.14 + j4.32 \\ &= 7.51\angle 35^\circ \text{ V}\end{aligned}$$

The impedance of the capacitor is given by

$$-j\frac{1}{2C} = \frac{\mathbf{V}_c(\omega)}{\mathbf{I}(\omega)} = \frac{7.51\angle 35^\circ}{1.59\angle 125^\circ} = 4.72\angle -90^\circ$$

Solving for C gives

$$C = \frac{-j}{2\left(4.72\angle -90^\circ\right)} = \frac{1\angle -90^\circ}{2\left(4.72\angle -90^\circ\right)} = 0.106\,\text{F}$$

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EXAMPLE 10.4-2 AC Circuits in the Frequency Domain

The input to the ac circuit shown in Figure 10.4-7 is the source voltage

$$v_S(t) = 48\cos(500t + 75°)\ \text{V}$$

Determine the voltage $v(t)$.

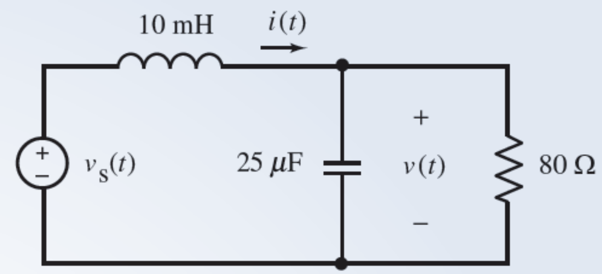

FIGURE 10.4-7 The ac circuit in Example 10.4-2.

Solution

The input frequency is $\omega = 500$ rad/s. The impedances of the capacitor and inductor are

$$\mathbf{Z}_C(\omega) = \frac{1}{j\omega C} = \frac{1}{j500\left(25\times 10^{-6}\right)} = \frac{80}{j} = -j80\ \Omega,$$

and

$$\mathbf{Z}_L(\omega) = j\omega L = j500(0.1) = j50\ \Omega$$

$$Z_C \parallel R = \frac{(-j80)80}{-j80+80} = 40 - j40$$

$$V = \frac{Z_C \parallel R}{Z_L + Z_C \parallel R}V_S = \frac{40 - j40}{j50 + (40 - j40)}V_S = \frac{12 - j20}{17}V_S$$

$$= (1.372\angle -59.04°)(48\angle 75°) = 65.9\angle 15.96°$$