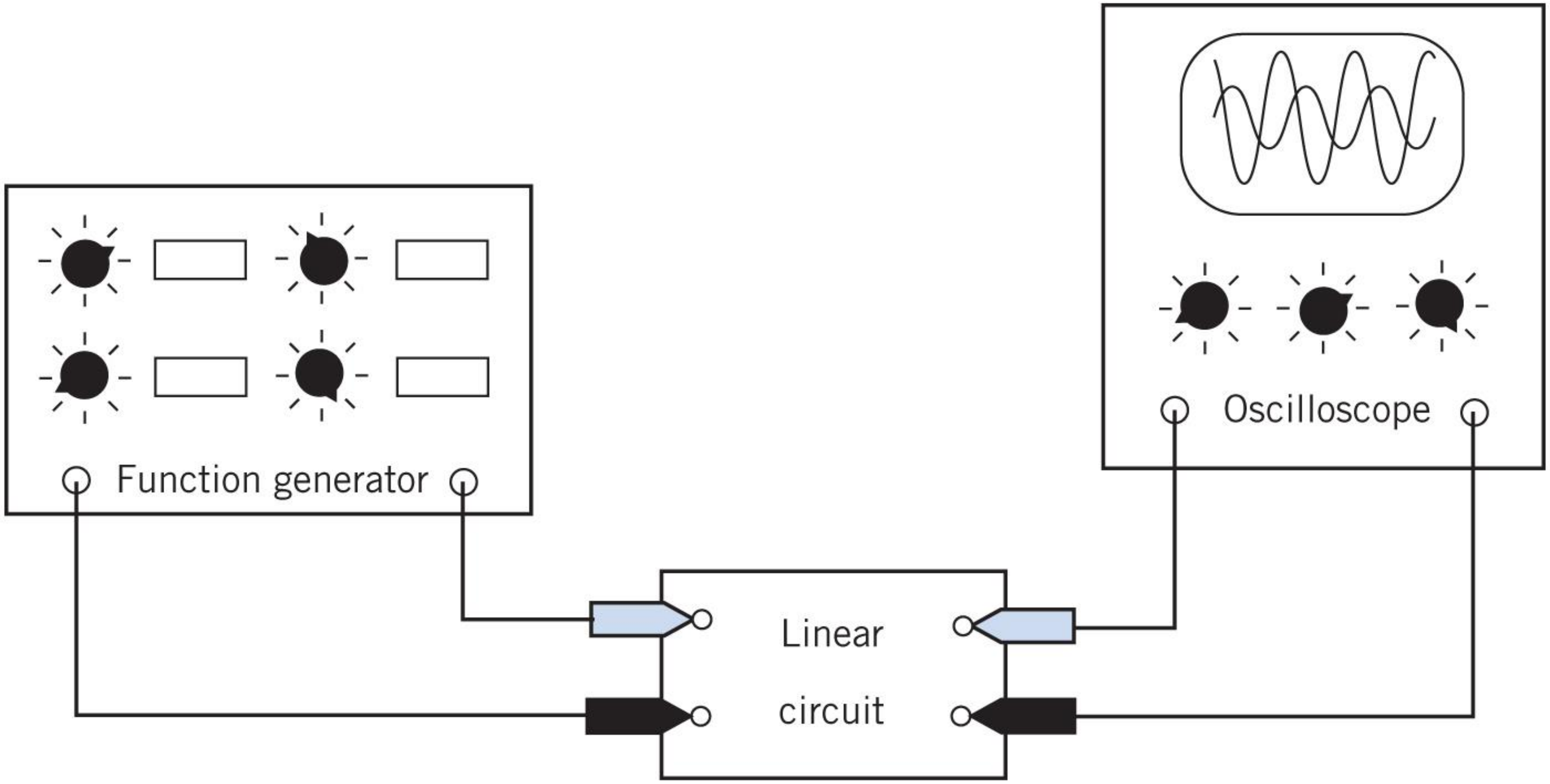
Frequency Response

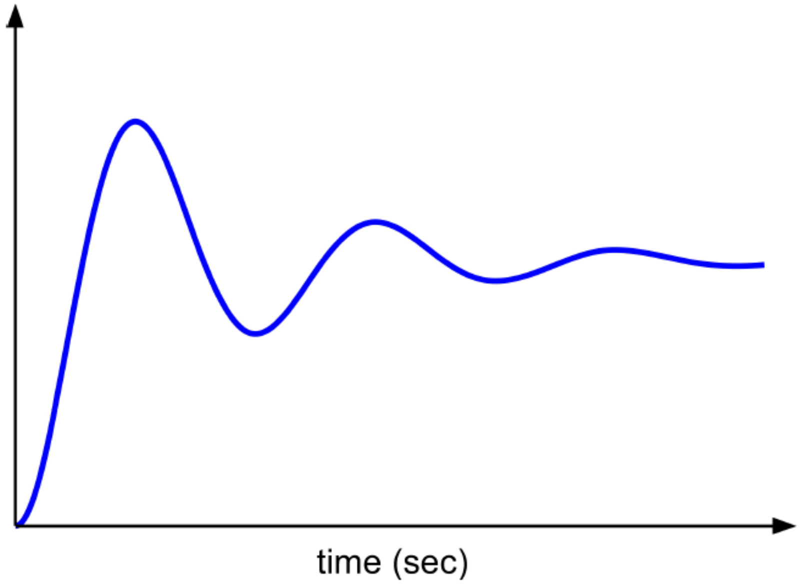
RC & RLC Circuits

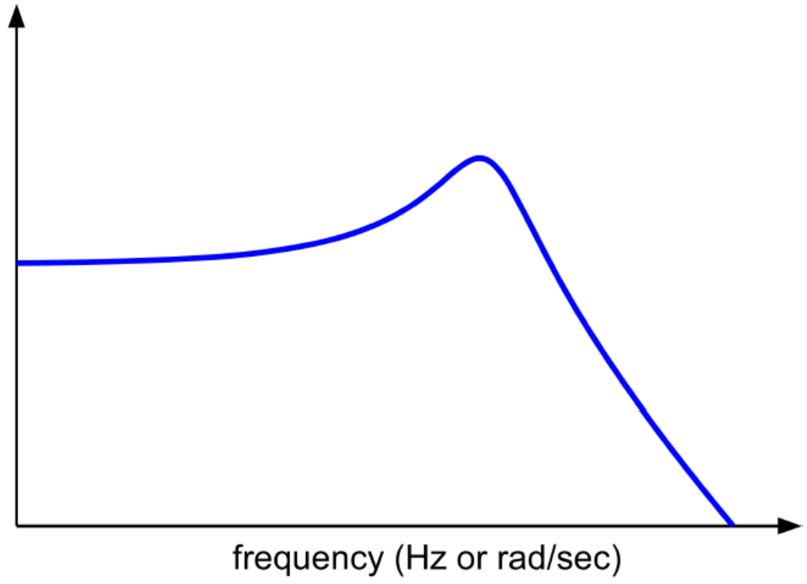
Measuring Responses

- Define input and output
- Time response and frequency response

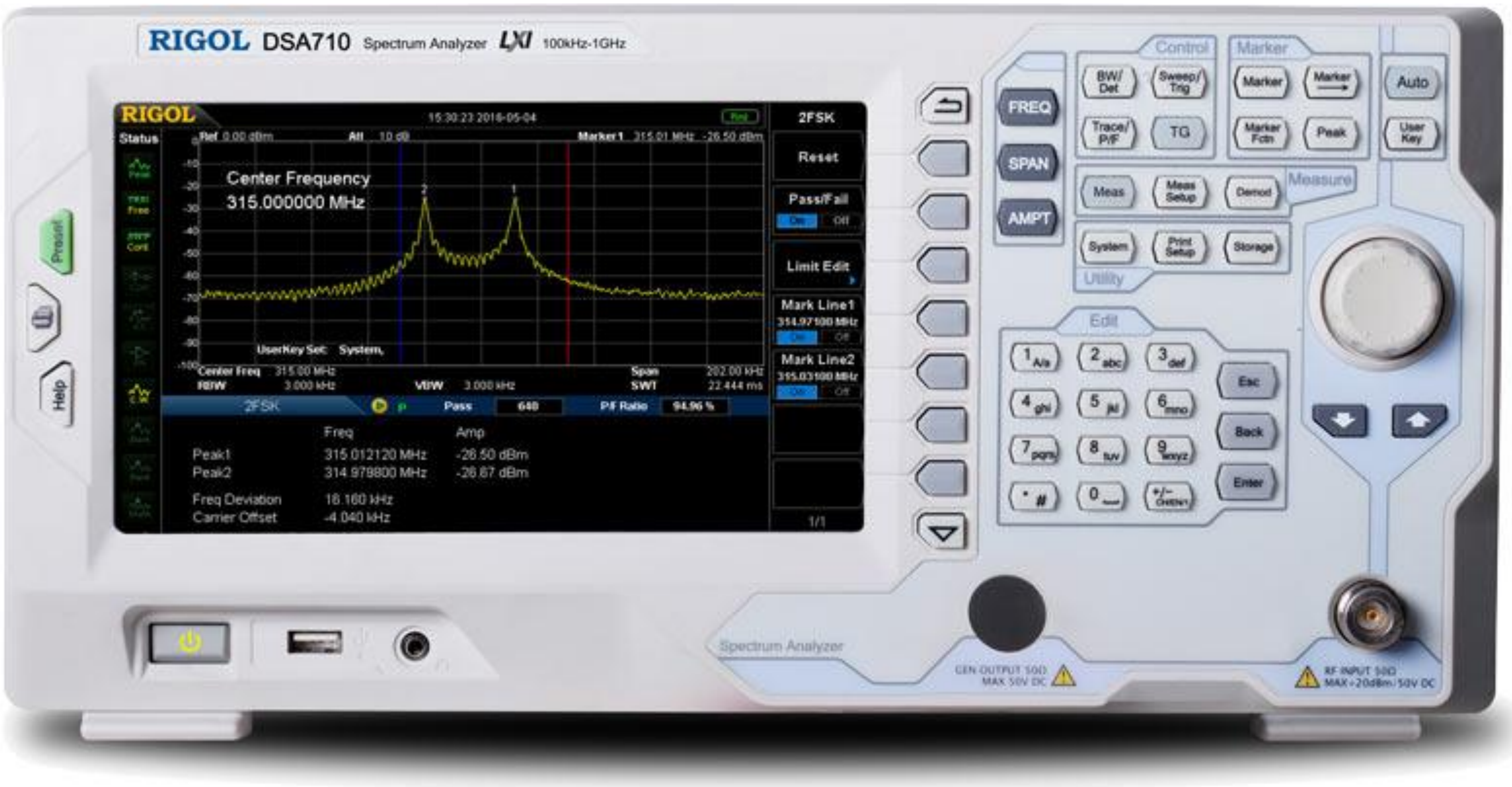


Time Response and Frequency Response



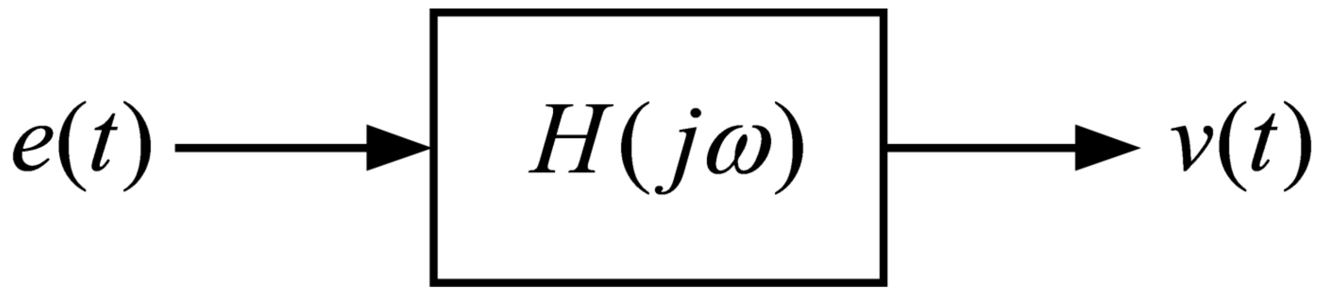


Spectrum Analyzer



Frequency Response

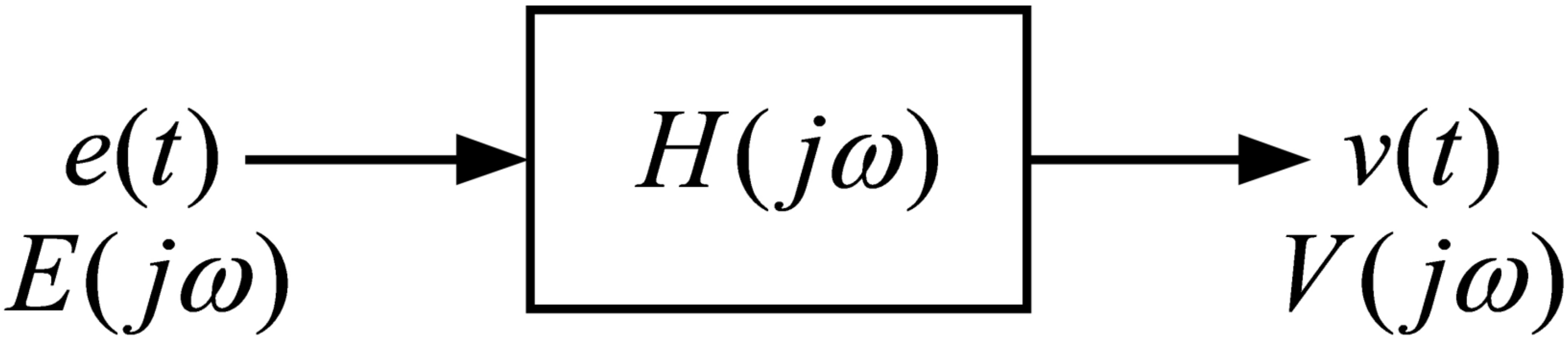
- Sinusoidal Input, Steady State



$$e(t) = A\cos(\omega t)$$

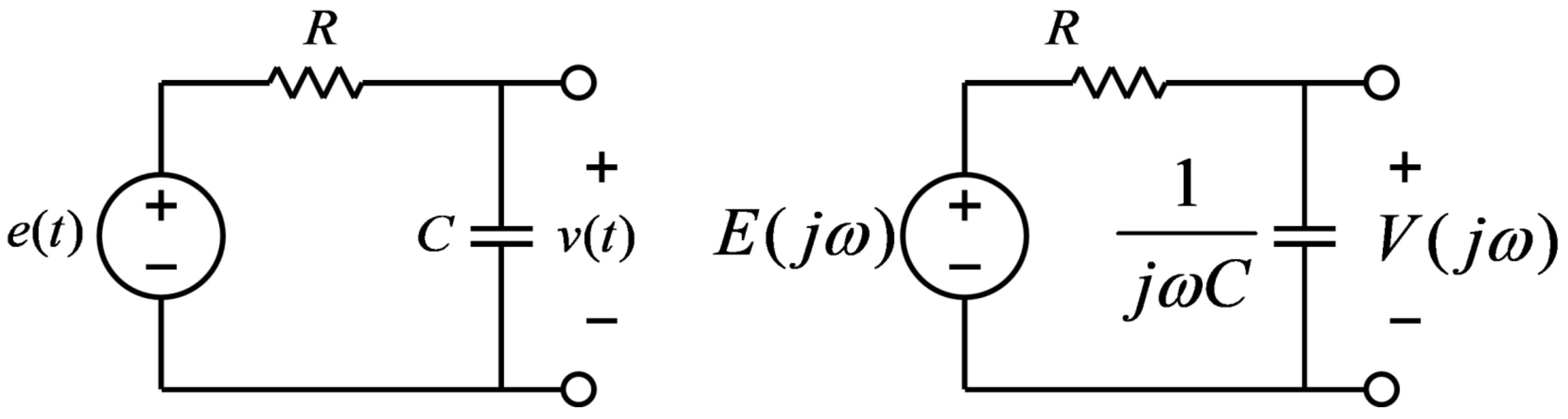
$$v(t) = A\left[\qquad\quad\right]\cos\left(\omega t + \left[\qquad\quad\right]\right)$$

Network Function



$$H(j\omega) = \frac{V(j\omega)}{E(j\omega)}$$

Network Function



$$V(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} E(j\omega) = \frac{1}{j\omega RC + 1} E(j\omega)$$

$$H(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{1}{j\omega RC + 1}$$

Network Function

$$V(j\omega) = \frac{V(j\omega)}{E(j\omega)} E(j\omega) = H(j\omega)E(j\omega)$$

$$= \left(\left|H(j\omega)\right| \angle H(j\omega)\right) E(j\omega)$$

$$e(t) = A\cos\omega t \Rightarrow E(j\omega) = A\angle 0^\circ$$

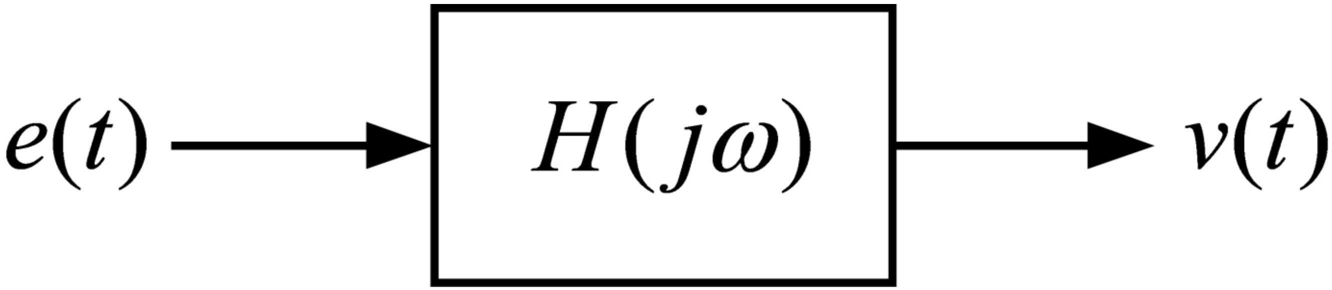
$$V(j\omega) = \left(\left|H(j\omega)\right| \angle H(j\omega)\right) A\angle 0^\circ$$

$$= A\left|H(j\omega)\right| \angle H(j\omega)$$

$$v(t) = A\left|H(j\omega)\right| \cos\left(\omega t + \angle H(j\omega)\right)$$

Frequency Response

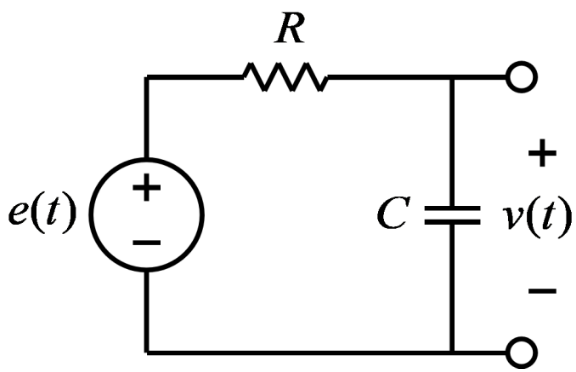
- Sinusoidal Input, Steady State

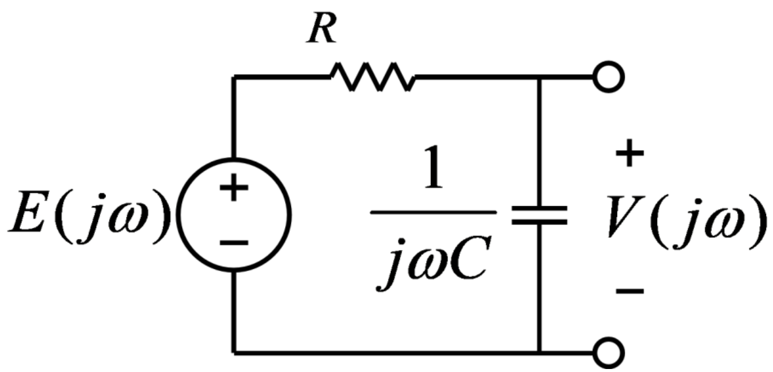


$$e(t) = A\cos(\omega t)$$

$$v(t) = A\left|H(j\omega)\right|\cos\left(\omega t + \angle H(j\omega)\right)$$

Low Pass RC Circuit





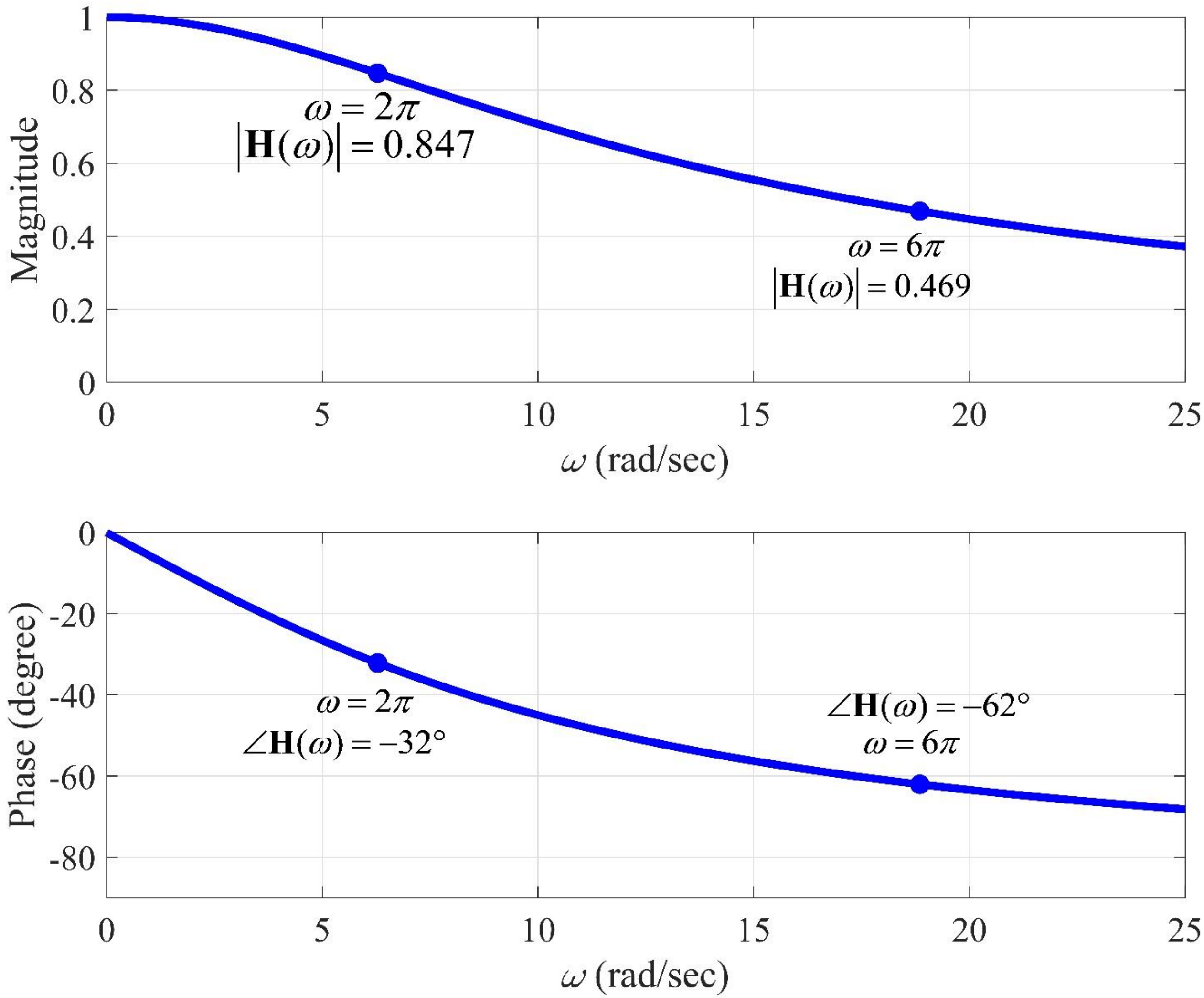
$$H(j\omega)=\frac{V(j\omega)}{E(j\omega)}=\frac{1}{j\omega RC+1}$$

$$\left|H(j\omega)\right|=\left|\frac{1}{j\omega RC+1}\right|=\frac{1}{\sqrt{(\omega RC)^2+1}}$$

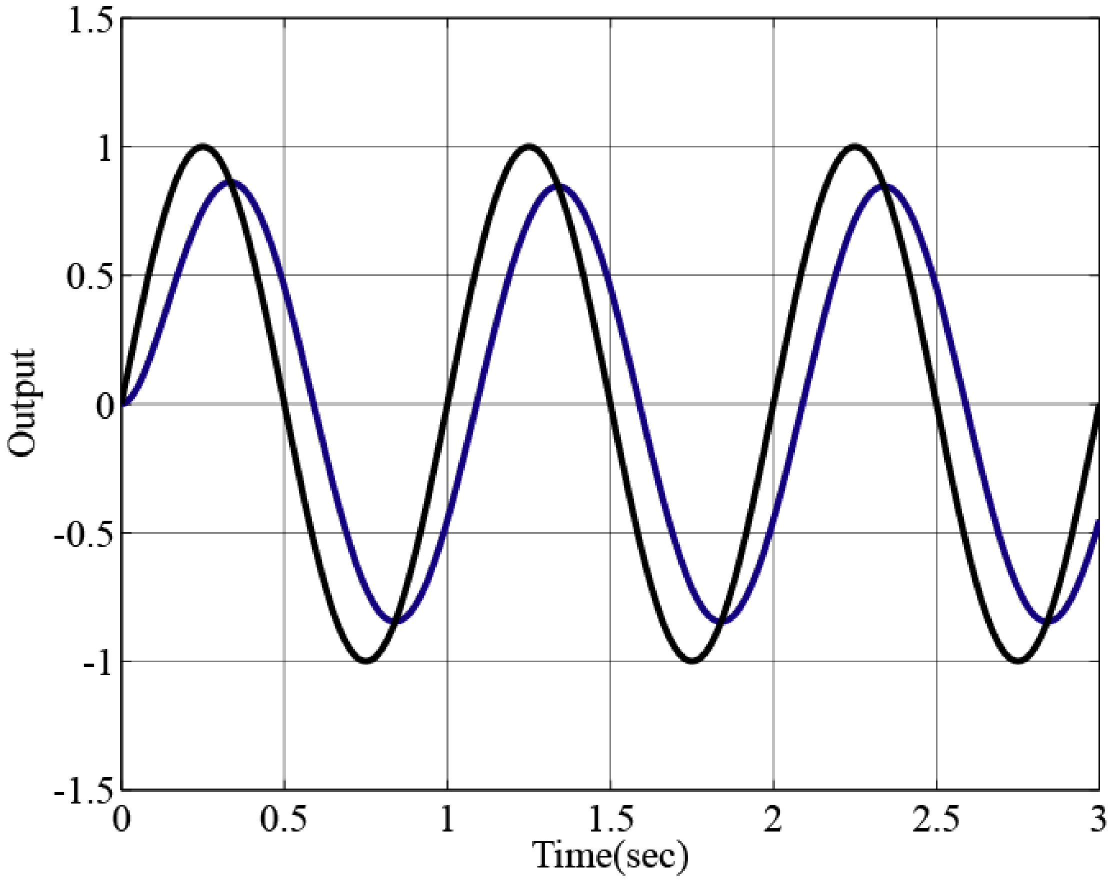
$$\angle H(j\omega)=\angle\left(\frac{1}{j\omega RC+1}\right)=\angle(1)-\angle(j\omega RC+1)=-\tan^{-1}(\omega RC)$$

$$\text{Note}:\angle(a+jb)=\tan^{-1}\frac{b}{a}$$

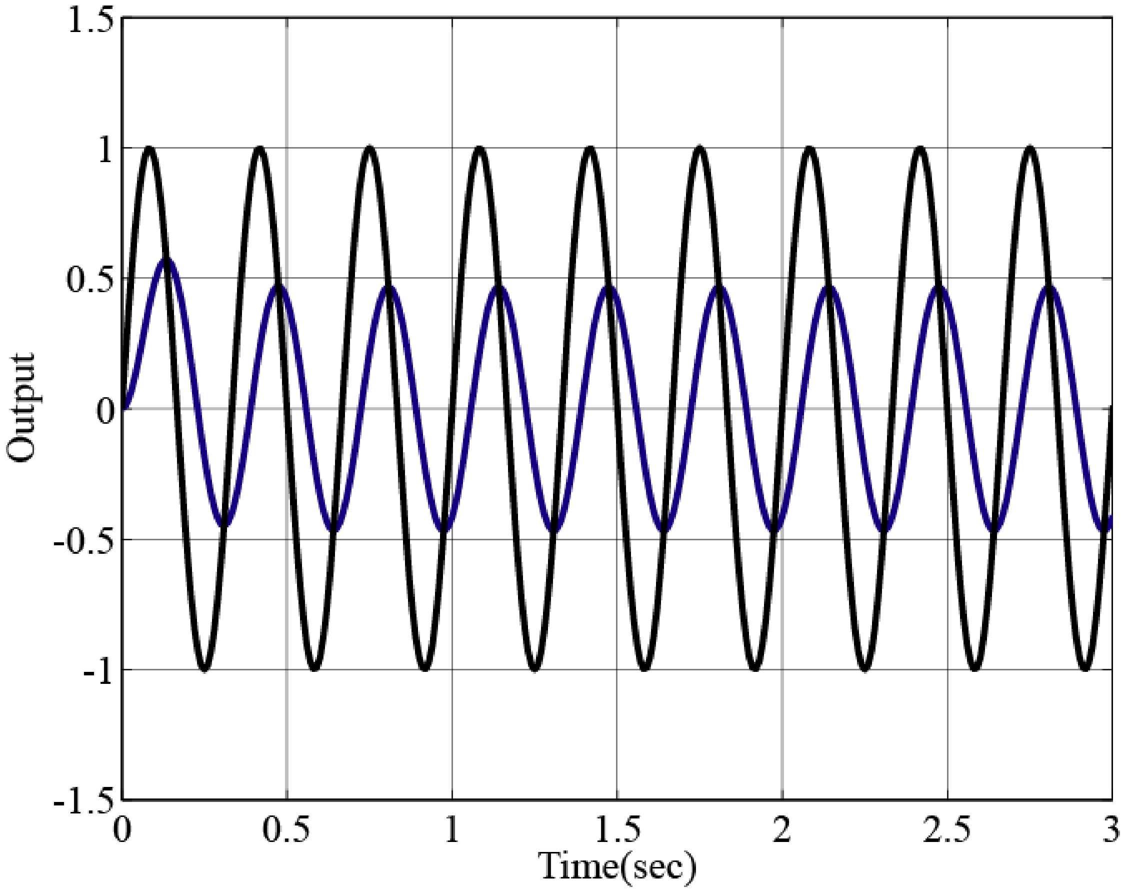
$$R = 100\Omega, C = 0.001F$$



1Hz

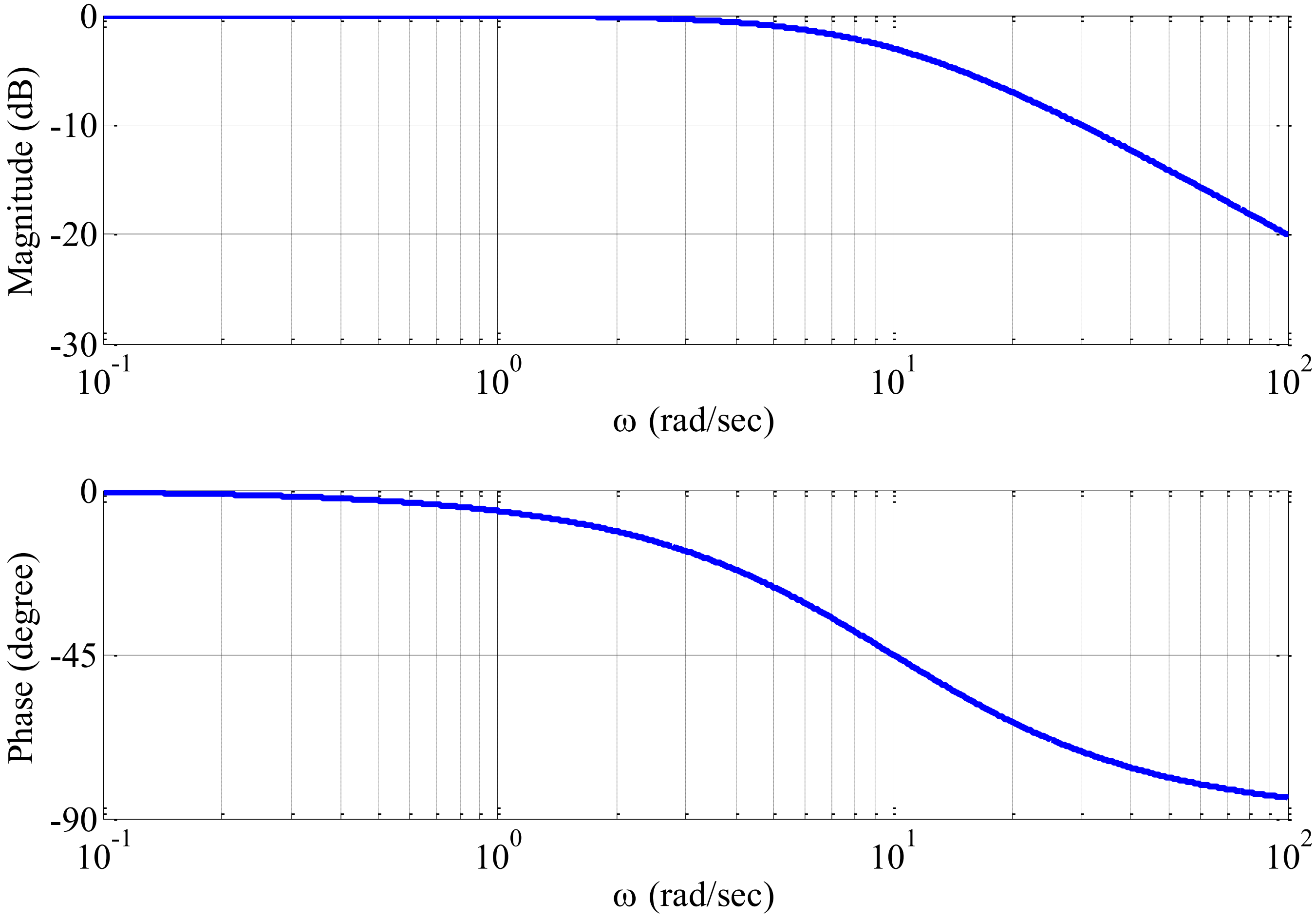


3Hz



Bode Plot

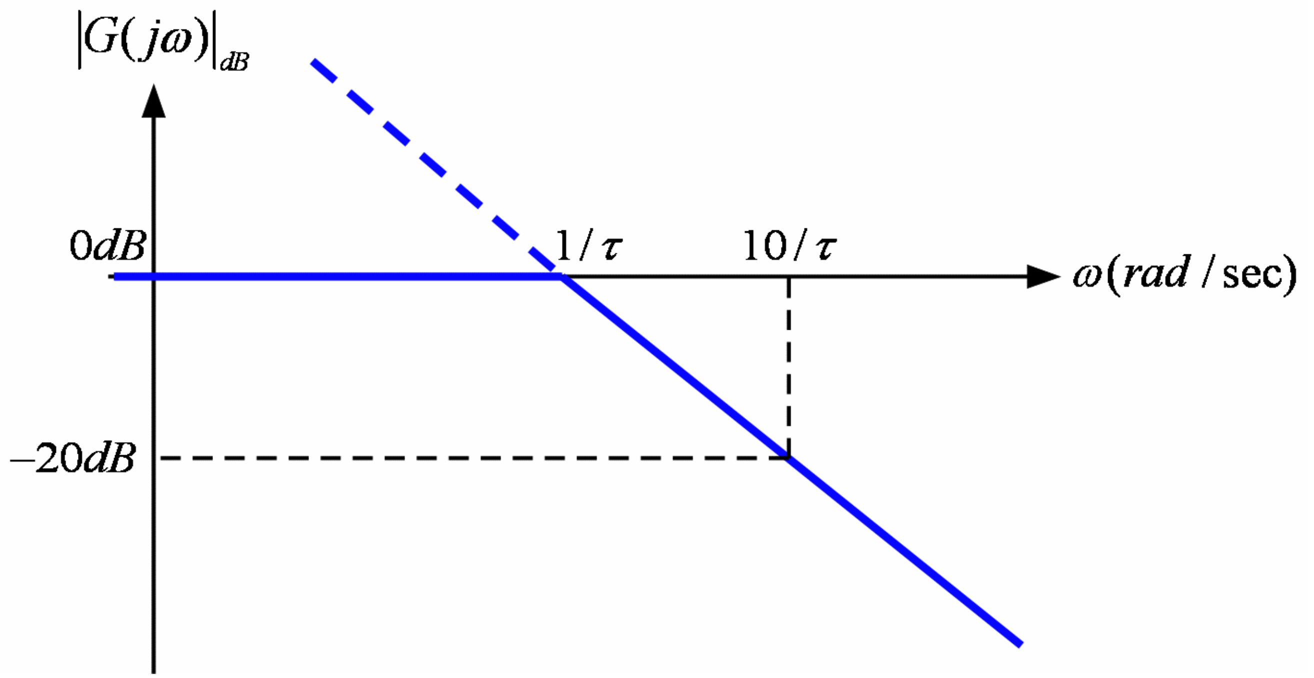
- dB(deci Bell): 20log(magnitude)



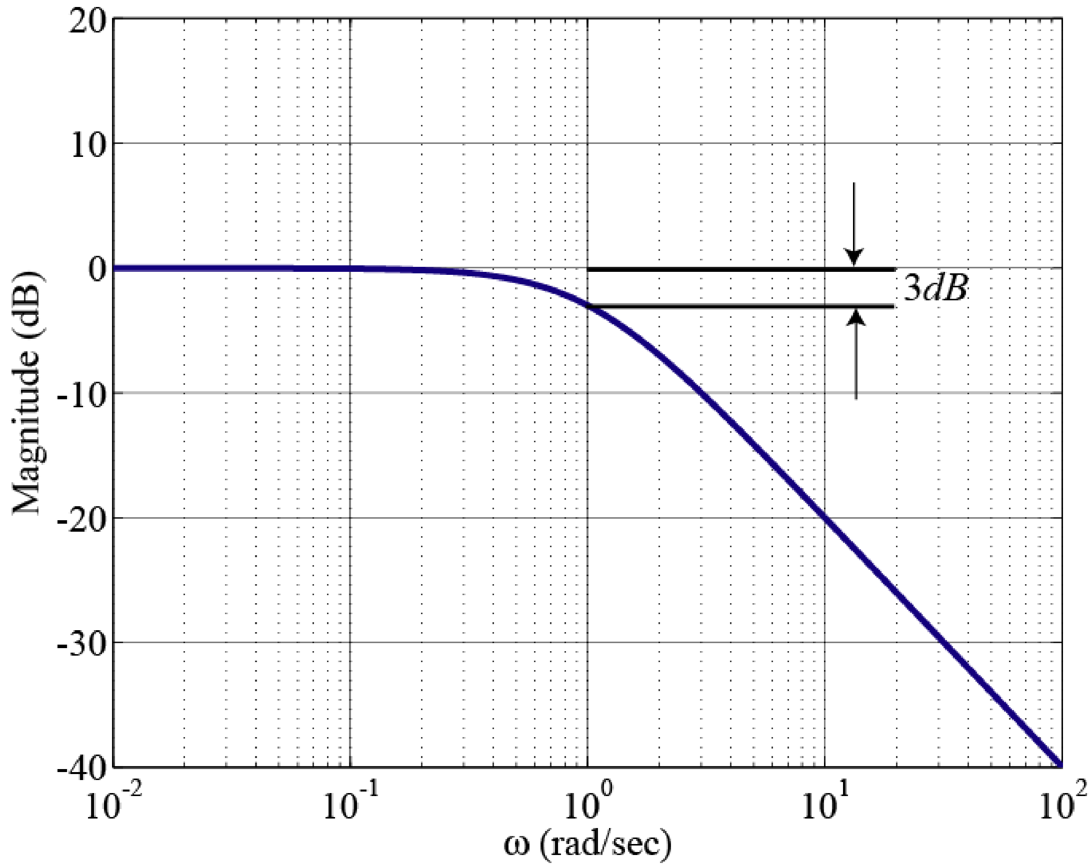
$1/(s\tau+1)$

$$\left|\frac{1}{j\omega\tau+1}\right|_{dB} = -20\log_{10}\left|j\omega\tau+1\right| \approx -20\log_{10}1 = 0 \quad \omega \ll 1/\tau$$

$$\left|\frac{1}{j\omega\tau+1}\right|_{dB} = -20\log_{10}\left|j\omega\tau+1\right| \approx -20\log_{10}\left|\omega\tau\right| \quad \omega \gg 1/\tau$$



$1/\left(s\tau+1\right)$

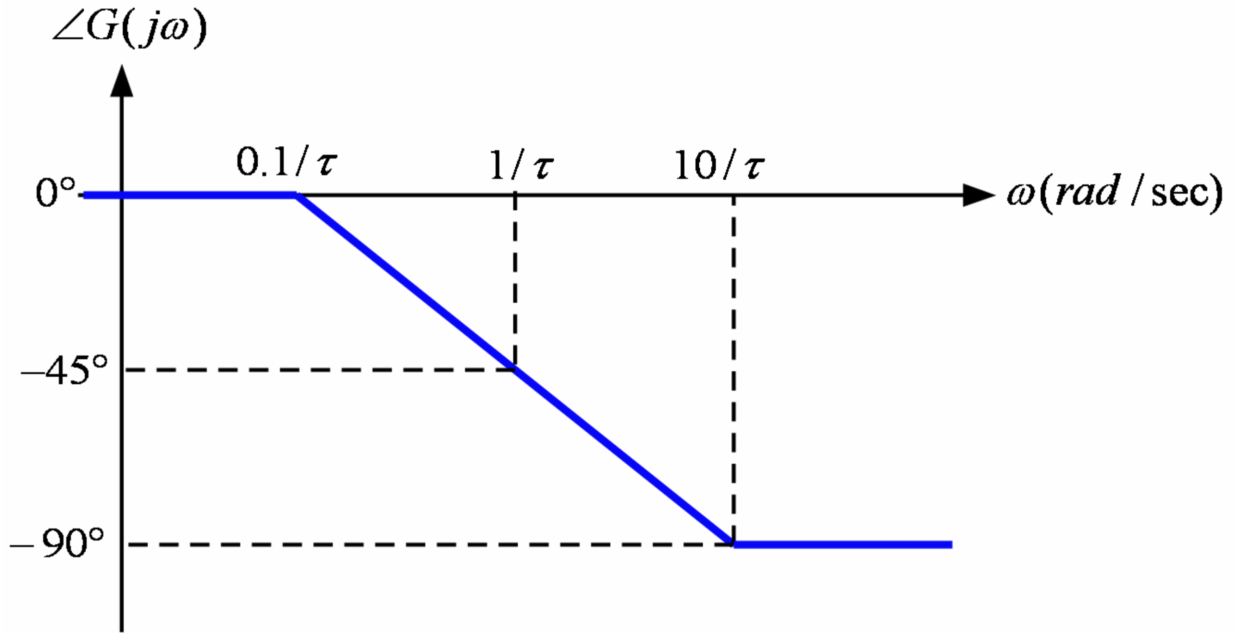


$1/(s\tau+1)$

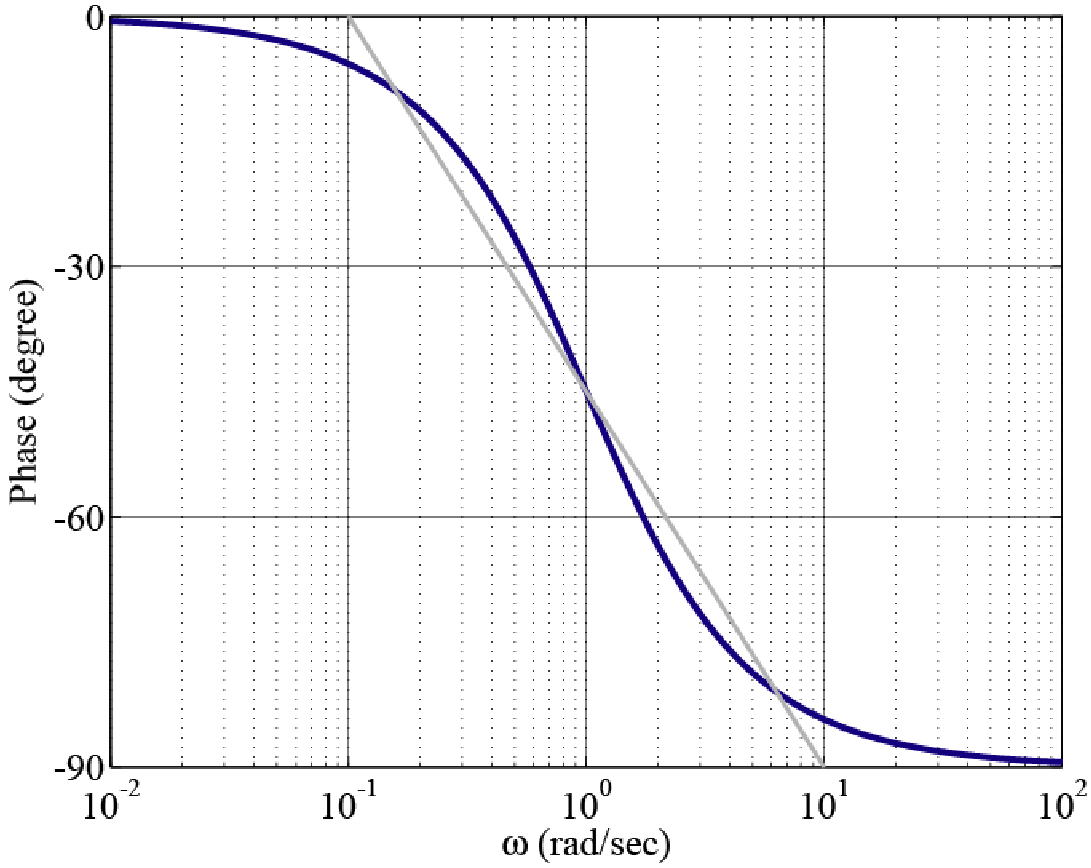
$$\angle\left(\frac{1}{j\omega\tau+1}\right)\approx-\angle 1=0^\circ \qquad \omega\ll 1/\tau$$

$$\angle\left(\frac{1}{j\omega\tau+1}\right)=-\angle(j+1)=-45^\circ \qquad \omega=1/\tau$$

$$\angle\left(\frac{1}{j\omega\tau+1}\right)\approx-\angle(j\omega\tau)=-90^\circ \qquad \omega\gg 1/\tau$$



$1/(s\tau+1)$



Band Width, Cut Off Frequency

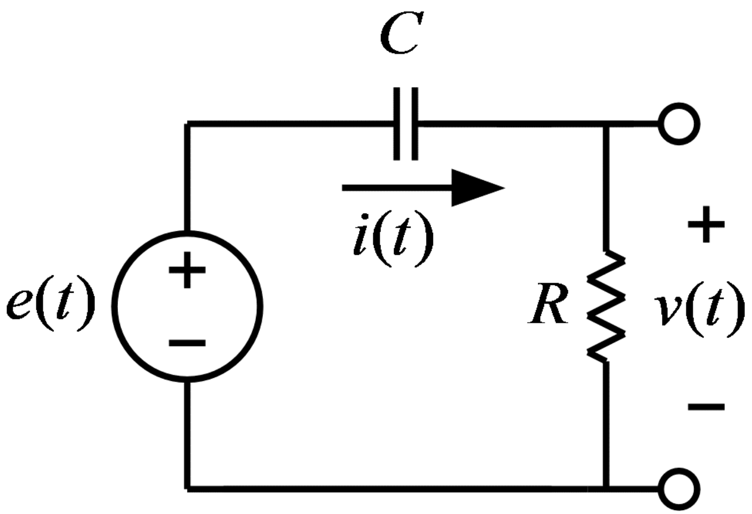
$$G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{1}{j\omega RC + 1}$$

$$\left|G(j\omega)\right| = \left|\frac{1}{j\omega RC + 1}\right| = \frac{1}{\sqrt{(\omega RC)^2 + 1}}$$

$$\omega RC = 1 \Rightarrow \left|G(j\omega)\right| = \left|\frac{1}{j+1}\right| = \frac{1}{\sqrt{2}} \Rightarrow 20\log_{10}\frac{1}{\sqrt{2}} = -3dB$$

$$\omega = \frac{1}{RC}$$

High Pass RC Circuit



$$V(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} E(j\omega) = \frac{j\omega RC}{j\omega RC + 1} E(j\omega)$$

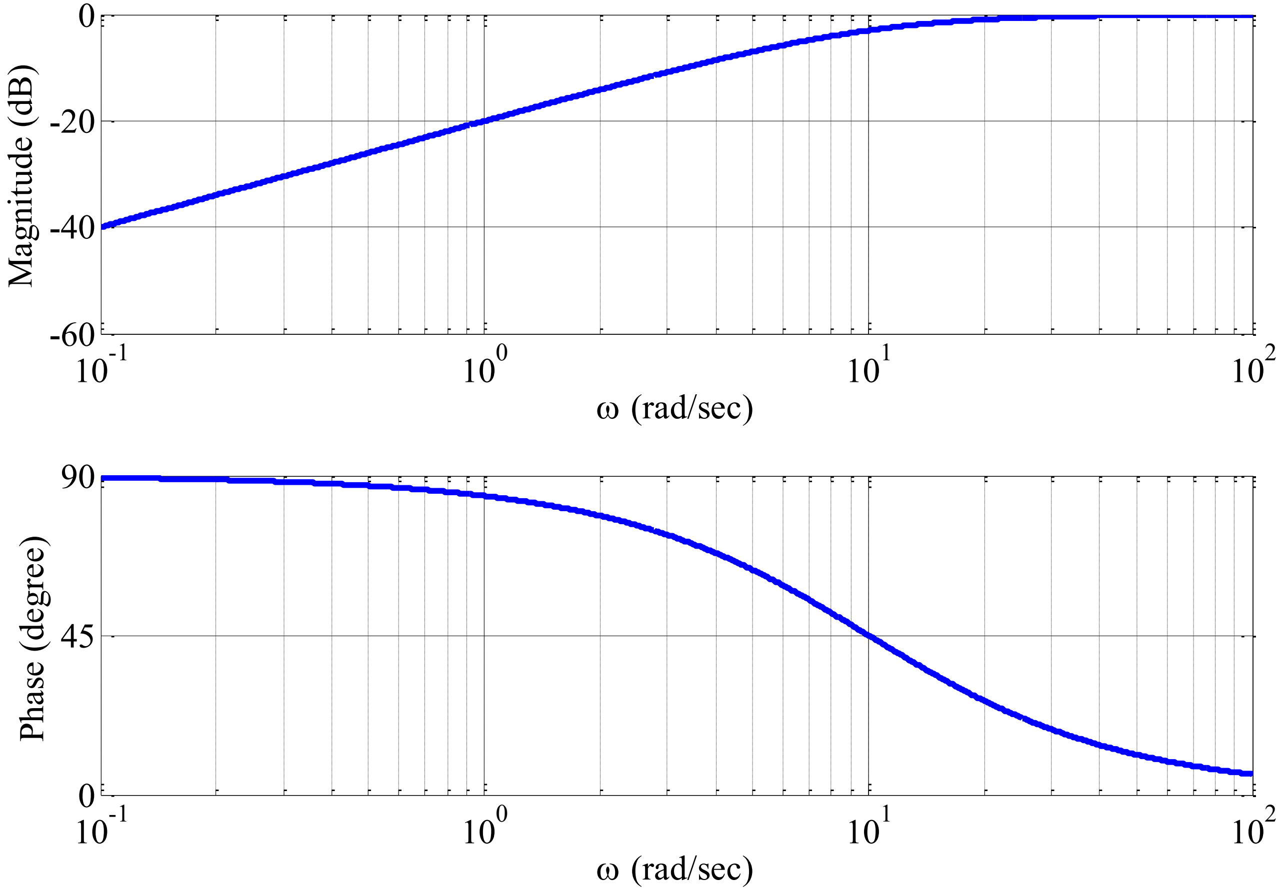
$$G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{j\omega RC}{j\omega RC + 1}$$

$$G(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{j\omega RC}{j\omega RC + 1}$$

$$\left|G(j\omega)\right| = \left|\frac{j\omega RC}{j\omega RC + 1}\right| = \frac{\omega RC}{\sqrt{\left(\omega RC\right)^2 + 1}}$$

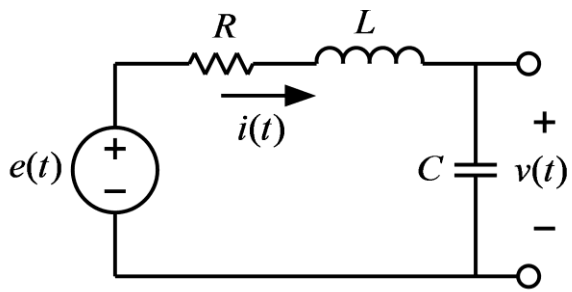
$$\measuredangle G(j\omega) = \measuredangle\left(\frac{j\omega RC}{j\omega RC + 1}\right) = 90^\circ - \tan^{-1}\left(\omega RC\right)$$

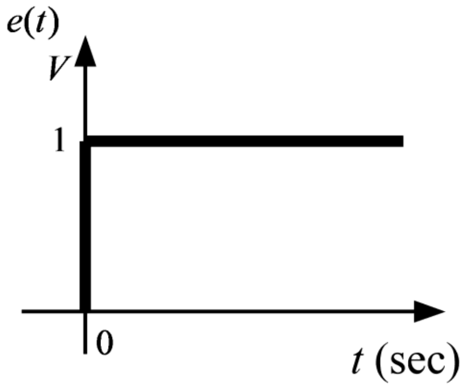
Bode Plot



0
-20
-40
-60
Magnitude (dB)
10^-1
10^0
10^1
10^2
ω (rad/sec)
90
45
0
Phase (degree)
10^-1
10^0
10^1
10^2
ω (rad/sec)

RLC Circuit





$$LC\frac{d^2v(t)}{dt^2}+RC\frac{dv(t)}{dt}+v(t)=e(t)$$

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0, s^2+2\alpha s+\omega_0^2=0$$

$$\alpha=\frac{R}{2L}, \omega_0=\frac{1}{\sqrt{LC}}$$

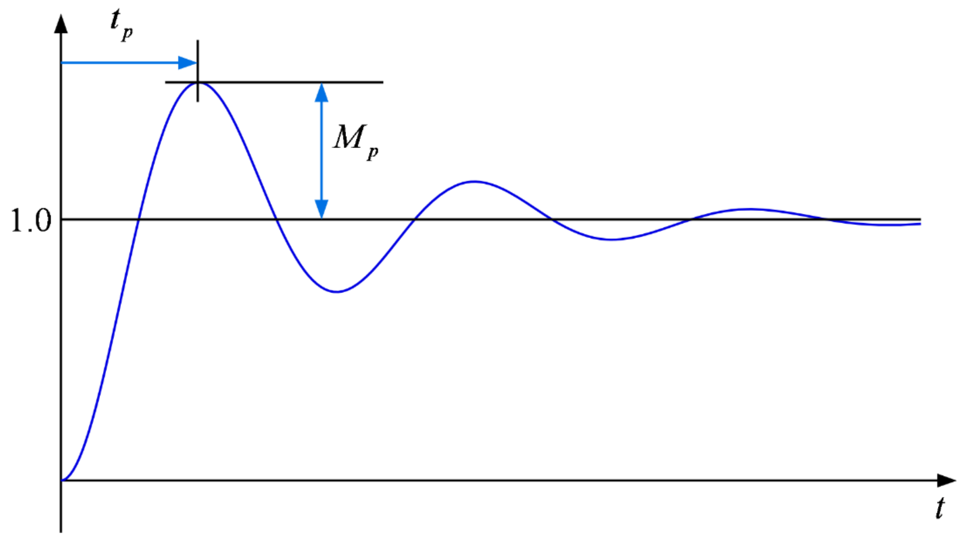
$$s^2+2\zeta\omega_0 s+\omega_0^2=0,$$

$$\text{damping ratio:} \zeta=\frac{\alpha}{\omega_0}, \alpha=\zeta\omega_0$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

$$\zeta < 1: s = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2}$$

$$v(t) = -\frac{1}{\sqrt{1-\left(\frac{\alpha}{\omega_0}\right)^2}} e^{-\alpha t}\cos\left(\omega_d t - \phi\right) + 1, \quad t \geq 0$$



Example 13.3-4 Network Function with Complex Poles

The network function of a second-order low-pass filter has the form

$$H(\omega) = \frac{k}{\left(j\frac{\omega}{\omega_0}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_0}\right) + 1}$$

$$\mathbf{H}(\omega) = \frac{k\,\omega_0^2}{(j\omega)^2 + j2\zeta\omega_0\omega + \omega_0^2}$$

This network function depends on three parameters: the dc gain k; the corner frequency ω_0; and the damping ratio ζ. For convenience, we consider the case where $k = 1$. Then, using $j^2 = -1$, we can write the network function as

$$\mathbf{H}(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j2\zeta\omega_0\omega}$$

Determine the asymptotic magnitude Bode plot of the second-order low-pass filter when the dc gain is 1.

Solution

The denominator of $\mathbf{H}(\omega)$ contains a new factor, one that involves ω^2. The asymptotic Bode plot is based on the approximation

$$\left(\omega_0^2 - \omega^2\right) + j2\zeta\omega_0\omega \cong \begin{cases} \omega_0^2 & \omega < \omega_0 \\ -\omega^2 & \omega > \omega_0 \end{cases}$$

Using this approximation, we can express $\mathbf{H}(\omega)$ as

$$\mathbf{H}(\omega) \cong \begin{cases} 1 & \omega < \omega_0 \\ -\dfrac{\omega_0^2}{\omega^2} & \omega > \omega_0 \end{cases}$$

The logarithmic gain is

$$20\log_{10}|\mathbf{H}(\omega)| \cong \begin{cases} 0 & \omega < \omega_0 \\ 40\log_{10}\omega_0 - 40\log_{10}\omega & \omega > \omega_0 \end{cases}$$

The asymptotic magnitude Bode plot is shown in Figure 13.3-10. The actual magnitude Bode plot and the actual phase Bode plot are shown in Figure 13.3-11. The asymptotic Bode plot is a good approximation to the actual Bode plot when $\omega \ll \omega_0$ or $\omega \gg \omega_0$. Near $\omega = \omega_0$, the asymptotic Bode plot deviates from the actual Bode plot. At $\omega = \omega_0$, the value of the asymptotic Bode plot is 0 dB, whereas the value of the actual Bode plot is

$$\mathbf{H}(\omega_0) = \frac{1}{2\zeta}$$

As this equation and Figure 13.3-11 both show, the deviation between the actual and asymptotic Bode plot near $\omega = \omega_0$ depends on ζ. The frequency ω_0 is called the *corner frequency*. The slope of the asymptotic Bode plot decreases by 40 dB/decade as the frequency increases past $\omega = \omega_0$. In terms of the asymptotic Bode plot, the denominator of this network function acts like two poles at $p = \omega_0$. If this factor were to appear in the numerator of a network function, it would act like two zeros at $z = \omega_0$. The slope of the asymptotic Bode plot would increase by 40 dB/decade as the frequency increased past $\omega = \omega_0$.

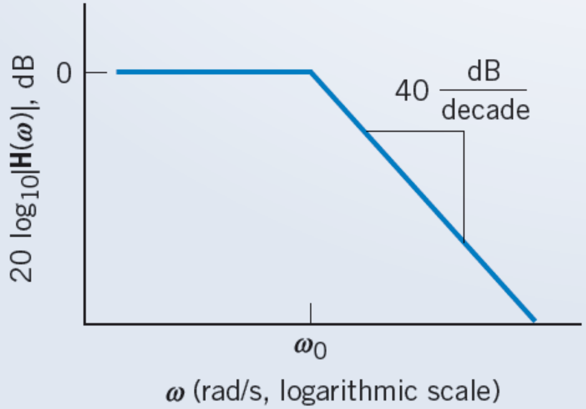


FIGURE 13.3-10 The asymptotic magnitude Bode plot of the second-order low-pass filter when the dc gain is 1.

$$1/\left[\left(j\omega/\omega_0\right)^2+\left(2\zeta\, j\omega/\omega_0\right)+1\right]$$

$$\left[\left(j\omega/\omega_0\right)^2+\left(j2\zeta\omega/\omega_0\right)+1\right]$$

$$=\left[\left(j\omega/\omega_0\right)+\zeta+\sqrt{\zeta^2-1}\right]\left[\left(j\omega/\omega_0\right)+\zeta-\sqrt{\zeta^2-1}\right]$$

real root: $1<\zeta$

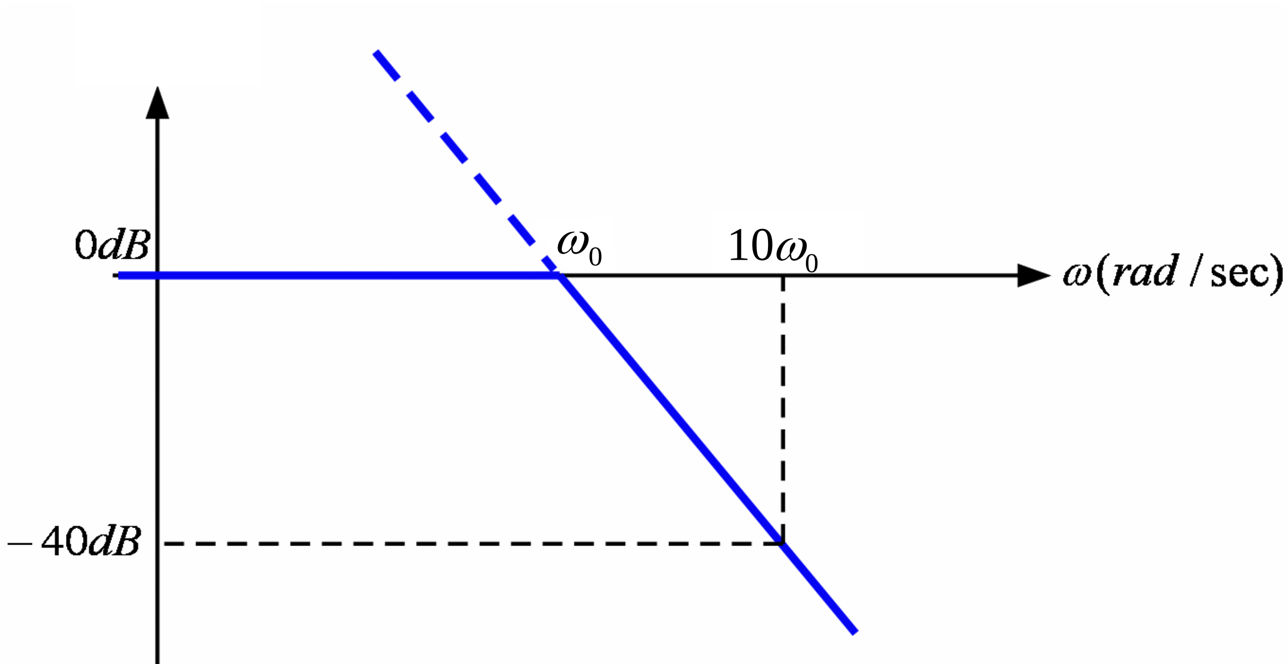
double root: $\zeta=1$

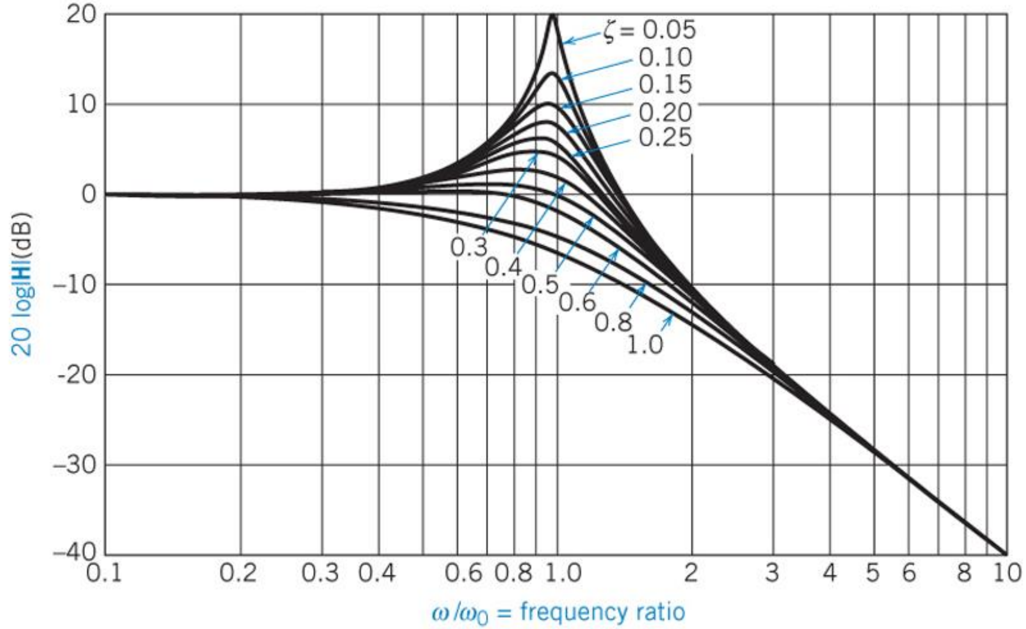
complex root: $\zeta<1$

$$1/\left[\left(j\omega/\omega_0\right)^2+\left(2\zeta j\omega/\omega_0\right)+1\right]$$

$$\left|\frac{1}{\left(j\omega/\omega_0\right)^2+\left(2\zeta j\omega/\omega_0\right)+1}\right|_{dB} \approx -20\log_{10}1=0 \qquad \omega \ll \omega_0$$

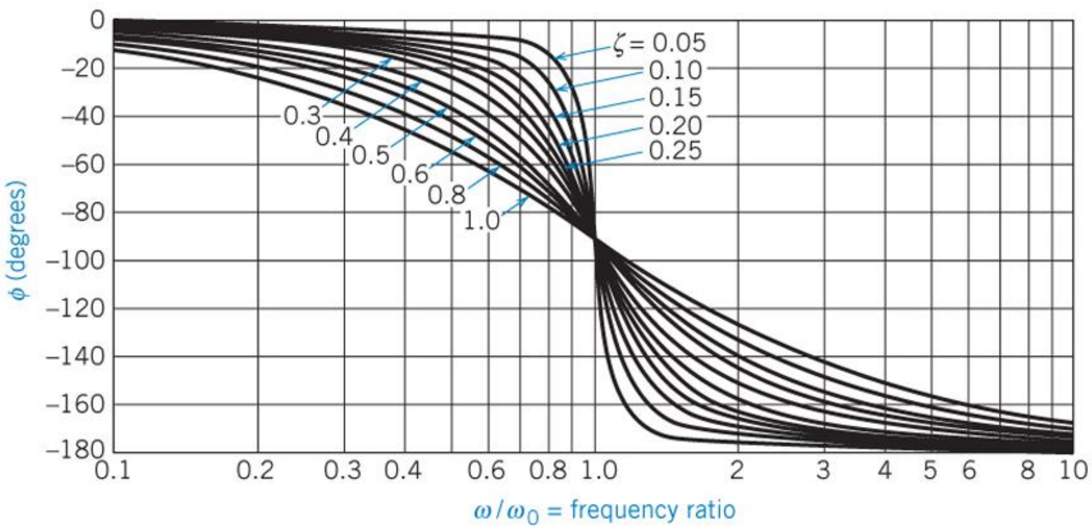
$$\left|\frac{1}{\left(j\omega/\omega_0\right)^2+\left(2\zeta j\omega/\omega_0\right)+1}\right|_{dB} \approx -20\log_{10}\left|\left(\omega/\omega_0\right)^2\right| \qquad \omega \gg \omega_0$$





20 log|H|(dB)
ζ = 0.05
0.10
0.15
0.20
0.25
0.3
0.4
0.5
0.6
0.8
1.0
20
10
0
−10
-20
−30
−40
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2
3
4
5
6
8
10
ω/ω0 = frequency ratio

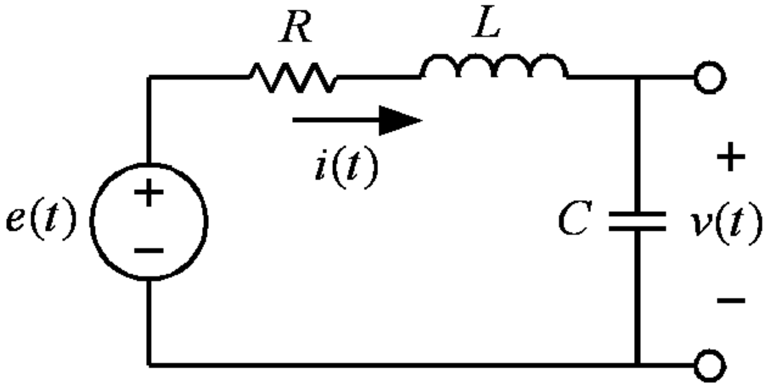
(a)



ϕ (degrees)
ζ = 0.05
0.10
0.15
0.20
0.25
0.3
0.4
0.5
0.6
0.8
1.0
0
−20
−40
−60
−80
−100
−120
−140
−160
−180
0.1
0.2
0.3
0.4
0.6
0.8
1.0
2
3
4
5
6
8
10
ω/ω0 = frequency ratio

(b)

Low Pass RLC Circuit



$$V(j\omega) = \frac{\dfrac{1}{j\omega C}}{j\omega L + R + \dfrac{1}{j\omega C}} E(j\omega) = \frac{1}{(j\omega)^2 LC + (j\omega)RC + 1} E(j\omega)$$

Low Pass RLC Circuit

$$V(j\omega) = \frac{\frac{1}{j\omega C}}{j\omega L + R + \frac{1}{j\omega C}} E(j\omega) = \frac{1}{(j\omega)^2 LC + (j\omega)RC + 1} E(j\omega)$$

$$H(j\omega) = \frac{V(j\omega)}{E(j\omega)} = \frac{1}{(j\omega)^2 LC + (j\omega)RC + 1} = \frac{1/(LC)}{(j\omega)^2 + (R/L)(j\omega) + 1/(LC)}$$

$$H(j\omega) = \frac{\omega_0^2}{(j\omega)^2 + 2\alpha(j\omega) + \omega_0^2}$$

$$\alpha = \frac{R}{2L}, \omega_0 = \frac{1}{\sqrt{LC}}, \zeta = \frac{\alpha}{\omega_0}$$

$$R = 100\Omega, L = 100mH, C = 0.1\mu F$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-1}10^{-7}}} = 10^4$$

$$\alpha = \frac{R}{2L} = \frac{100}{2\times 10^{-1}} = 500, \zeta = \frac{500}{10000} = 0.05$$

$$|H(j\omega_0)| = \frac{1}{2\zeta} = 10 = 20dB$$