RLC Circuits

2nd Order Differential Equation

$$\ddot{x} + 3\dot{x} + 2x = 0$$

$$x = Ke^{\lambda t}, \dot{x} = K\lambda e^{\lambda t}, \ddot{x} = K\lambda^2 e^{\lambda t}$$

$$\ddot{x} + 3\dot{x} + 2x = K\lambda^2 e^{\lambda t} + 3K\lambda e^{\lambda t} + 2Ke^{\lambda t}$$

$$= K\left(\lambda^2 + 3\lambda + 2\right)e^{\lambda t} = 0$$

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\lambda = -1, -2$$

- Assume:

$$x = K_1 e^{-t} + K_2 e^{-2t}$$

- Initial Conditions: $x(0) = 1, \dot{x}(0) = 0$

$$x(0) = K_1 + K_2 = 1$$

$$\dot{x}(0) = -K_1 - 2K_2 = 0$$

$$K_1 = 2, K_2 = -1$$

$$x(t) = 2e^{-t} - e^{-2t}$$

Non-zero Input

$$\ddot{x} + 3\dot{x} + 2x = 1$$

$$x = Ke^{\lambda t} + K_3$$

$$\ddot{x} + 3\dot{x} + 2x = K\lambda^2 e^{\lambda t} + 3K\lambda e^{\lambda t} + 2Ke^{\lambda t} + 2K_3$$

$$= K\left(\lambda^2 + 3\lambda + 2\right)e^{\lambda t} + 2K_3 = 1$$

$$\lambda = -1, -2$$

$$K_3 = 0.5$$

$$x(0) = 0, \dot{x}(0) = 0$$

$$x(0) = K_1 + K_2 + 0.5 = 0$$

$$\dot{x}(0) = -K_1 - 2K_2 = 0$$

$$K_1 = -1, K_2 = 0.5$$

$$x = -e^{-t} + 0.5e^{-2t} + 0.5$$

Complex Roots

$$\ddot{x} + 2\dot{x} + 5x = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = -1 \pm j2$$

$$x = K_1 e^{(-1+j2)t} + K_2 e^{(-1-j2)t}$$

$$K_2 = K_1^*$$

$$x = K_1 e^{(-1+j2)t} + K_1^* e^{(-1-j2)t} = 2\,\mathrm{Re}\left\{K_1 e^{(-1+j2)t}\right\}$$

$$x(0) = 1, \dot{x}(0) = 0$$

$$x(0) = K_1 + K_1^* = 1$$

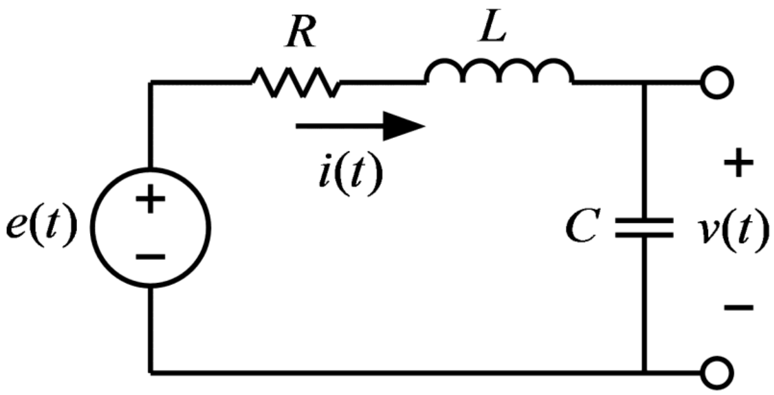
$$\dot{x}(0) = K_1(-1 + j2) + K_1^*(-1 - j2) = 0$$

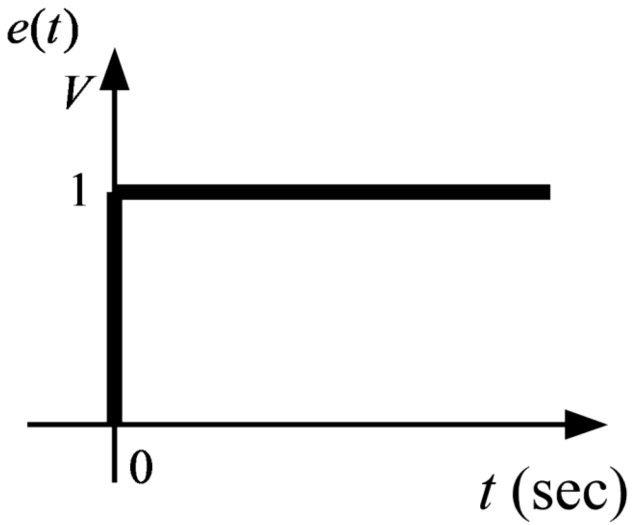
$$K_1 = 0.5 - j0.25$$

$$K_1^* = 0.5 + j0.25$$

$$x(t) = e^{-t}\left(\cos 2t + 0.5 \sin 2t\right)$$

RLC Circuit





$$Ri(t) + L\frac{di(t)}{dt} + v(t) = e(t)$$

$$i(t) = C\frac{dv(t)}{dt}$$

$$LC\frac{d^2v(t)}{dt^2} + RC\frac{dv(t)}{dt} + v(t) = e(t)$$

Characteristic Equation

$$LC\frac{d^2v(t)}{dt^2}+RC\frac{dv(t)}{dt}+v(t)=e(t)$$

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

$$s_1=-\frac{R}{2L}+\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

$$s_2=-\frac{R}{2L}-\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

Overdamped Case ($\alpha > \omega_0$)

From Eqs. (8.9) and (8.10), $\alpha > \omega_0$ implies $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$v(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + 1$$

which decays and approaches zero as t increases. Figure 8.9(a) illustrates a typical overdamped response.

Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{8.15}$$

$$v(t) = K_1 e^{-\alpha t} + K_2 t e^{-\alpha t} + 1$$

Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$, $C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \tag{8.22a}$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \tag{8.22b}$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*. Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is often called the *undamped natural frequency*, ω_d is called the *damped natural frequency*. The natural

$$v(t) = K_1 e^{-(\alpha - j\omega_d)t} + K_2 e^{-(\alpha + j\omega_d)t} + 1$$

$$= e^{-\alpha t}\left(K_1 e^{j\omega_d t} + K_2 e^{-j\omega_d t}\right) + 1$$

$$= e^{-\alpha t}\left[K_1\left(\cos\omega_d t + j\sin\omega_d t\right) + K_2\left(\cos\omega_d t - j\sin\omega_d t\right)\right] + 1$$

$$= e^{-\alpha t}\left[\left(K_1 + K_2\right)\cos\omega_d t + j\left(K_1 - K_2\right)\sin\omega_d t\right] + 1$$

$$v(0) = 0, \left.\frac{dv}{dt}\right|_{t=0} = 0$$

$$v(t) = -\frac{1}{\sqrt{1-\left(\frac{\alpha}{\omega_0}\right)^2}} e^{-\alpha t}\cos\left(\omega_d t - \phi\right) + 1, \quad t \geq 0$$

$$\phi = \tan^{-1}\frac{\left(\frac{\alpha}{\omega_0}\right)}{\sqrt{1-\left(\frac{\alpha}{\omega_0}\right)^2}}$$